

Efficient Risk Sharing with Limited Commitment and Storage*

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Abstract

We extend the model of risk sharing with limited commitment (Kocherlakota, 1996) by introducing both a public and a private (unobservable and/or non-contractible) storage technology. Positive public storage relaxes future participation constraints, hence it can improve risk sharing, contrary to the case where hidden income or effort is the deep friction. The characteristics of constrained-efficient allocations crucially depend on the storage technology's return. At the steady state, if the return on storage is (i) moderately high, both assets and the consumption distribution may remain time-varying; (ii) sufficiently high, assets converge almost surely to a constant and the consumption distribution is time-invariant; (iii) equal to agents' discount rate, perfect risk sharing is self-enforcing. Agents never have an incentive to use their private storage technology, i.e., Euler inequalities are always satisfied, at the constrained-efficient allocation of our model, while this is not the case without optimal public asset accumulation. Finally, we find that, in contrast with the limited-commitment model without storage, past income affects consumption growth negatively both in our model with storage and in data from Indian villages.

Keywords: risk sharing, limited commitment, hidden storage, dynamic contracts

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1 Introduction

The literature on incomplete markets either restricts asset trade exogenously, most prominently by allowing only a risk-free bond to be traded (Huggett, 1993; Aiyagari, 1994), or considers a deep friction which limits risk sharing endogenously. With private information as the friction, a few papers (Allen, 1985; Cole and Kocherlakota, 2001; Ábrahám, Koehne, and Pavoni, 2011) have integrated these two strands of literature by introducing a storage technology. This paper considers limited commitment (Thomas and Worrall, 1988; Kocherlakota, 1996), LC hereafter, and makes a similar contribution by introducing both a public and a private storage technology.

Our starting point is the two-sided lack of commitment framework of Kocherlakota (1996), the basic model hereafter. Agents face risk in that their share of a constant aggregate income is stochastic. Risk-sharing transfers have to be such that both agents are at least as well off as in autarky at each time and state of the world. The storage technology we introduce allows the planner and the agents to transfer resources from one period to the next and earn a net return r , $-1 \leq r \leq 1/\beta - 1$, where β is agents' subjective discount factor.¹ Storage by agents is hidden, i.e., unobservable and/or non-contractible, while an agent reverting to autarky is excluded from the returns of the publicly accumulated assets, an endogenous Lucas tree.²

The model of risk sharing with LC has been successfully applied to better understand the consumption of households in village economies (Ligon, Thomas, and Worrall, 2002; Laczó, 2015) and in the United States (Krueger and Perri, 2006), of spouses within a household (Mazzocco, 2007; Voena, 2015), as well as international business cycles (Kehoe and Perri, 2002). In these applications, agents are likely to have a way to transfer resources intertemporally, both jointly and privately. However, typically neither public nor hidden storage is considered.³

¹Note that with $r = -1$ we are back to the Kocherlakota (1996) setting.

²Krueger and Perri (2006) make the same assumption for their exogenous Lucas tree. Publicly stored assets may be protected by community enforcement (guards for public grain storage facilities in villages), by contracts (divorce law for couples), or by international organisations (for countries). Alternatively, one may think of an outside financial intermediary implementing public storage, as in our decentralisation, see Section 2.5. Note that Karaivanov and Townsend (2014) assume the presence of a financial intermediary as well, for Thai villages.

³A few papers have considered LC economies with an intertemporal technology, either pure storage or capital accumulation. In Marcet and Marimon (1992) and Kehoe and Perri (2002), the social planner allocates capital to the agents, which in turn increases their outside option. In Voena (2015) the share of assets given to each divorcee is determined by divorce laws. In contrast, in our model agents cannot expropriate the public assets upon default. In a risk-sharing framework with LC, Ligon, Thomas, and Worrall (2000) consider observable and contractible individual storage and no public storage. In their environment individual storage is used in equilibrium, in contrast to our framework. Krueger and Perri (2006) introduce public assets of a constant size and show that its presence increases the amount of risk sharing, as in our model. As opposed to our paper, they do not endogenise the size of the asset/Lucas tree. Finally, Ábrahám and Cárceles-Poveda

Our analysis with two agents is particularly relevant for applications where risk sharing is studied within a household, between ‘husband’ and ‘wife.’ In a recent survey, [Chiappori and Mazzocco \(2017\)](#) consider the LC model, possibly with household savings, as the most powerful model both to explain the intra-household allocation of consumption and leisure, and to study normative questions such as joint taxation of household income and assets. Papers which consider endogenous household savings typically assume that savings do not affect the distribution of consumption ([Mazzocco, 2007](#); [Lise and Yamada, 2015](#)). In our analysis saving may respond to consumption dispersion, and, in turn, consumption dispersion next period may respond to savings. Some recent papers take into account the joint dynamics of individual consumptions and household savings ([Mazzocco, Ruiz, and Yamaguchi, 2013](#); [Voena, 2015](#)). However, they do not provide any analytical characterisation or discuss quantitatively the relationship between household savings and within-household consumption dispersion.⁴

Another set of applications where our finite-number-of-agents assumption is especially relevant is village economies. Some villages may have only few households, hence sizeable idiosyncratic shocks may affect aggregate income and income inequality, which in turn may affect savings.⁵ This application is also important because there is a sizeable literature, starting from [Ligon, Thomas, and Worrall \(2002\)](#), to test the extent of informal risk sharing in villages using a limited-commitment framework. In addition, there is some evidence that public storage facilities exist and are used for risk-sharing purposes. In particular, [Bliss and Stern \(1982\)](#) provide a detailed study of an Indian village, Palanpur, where an institution called the Seed Store functions not unlike public storage in our model. It is a co-operative with shares owned by member villagers. They use the Seed Store to borrow seeds to plant, and then pay back after the harvest. In case of need, repayments can be postponed, hence it contains an insurance element. The punishment for long-term inability to repay is a loss of membership, or, access to the Seed Store.

We first add only public storage to the basic model. The characteristics of constrained-efficient allocations, such as steady-state asset and consumption dynamics, crucially depend on the return on storage. First, we show that public storage is used in equilibrium as long as its return is sufficiently high and risk sharing is partial in the basic model. Further,

(2006) introduce public capital and derive the recursive form of the problem, similar to ours, but then they focus on the decentralisation of the constrained-optimal allocation as a competitive equilibrium. None of these papers, or any other in the LC literature, to our knowledge, considers hidden storage.

⁴Note that these quantitative works on couples’ decisions allow for divorce along the equilibrium path due to ‘love’ or ‘match quality’ shocks. In this paper we limit our attention to sharing income shocks and to couples that stay together, and leave the analytical study of how saving incentives are affected by the possibility of divorce to future work.

⁵In fact, our analysis highlights new mechanisms in all contexts where income inequality varies over time.

if the return on storage is moderately high, assets remain stochastic and the consumption distribution varies over time in the long run. If the return on storage is sufficiently high, assets converge almost surely to a constant and the consumption distribution is time-invariant. Risk sharing remains partial as long as the storage technology is inefficient, i.e., $r < 1/\beta - 1$, and perfect risk sharing is self-enforcing at the steady state if the return on storage is equal to agents' discount rate.

To understand how public storage matters, note that limited commitment makes markets endogenously incomplete, i.e., individual consumptions vary across income realisations and over time. Market incompleteness combined with idiosyncratic risk implies that some (precautionary) savings/storage improves welfare at the individual level. A benevolent social planner inherits these motives and stores 'on behalf of' the agents. At the same time, higher public assets reduce default incentives, thereby reducing consumption dispersion and in turn the precautionary motive for saving. Further, agents would like to front-load consumption as long as $\beta(1+r) < 1$. Optimal public asset accumulation is determined by these conflicting forces. If $\beta(1+r) = 1$, it is optimal for the planner to fully complete the market by storage at the steady state. This is because the trade-off between imperfect insurance and an inefficient intertemporal technology is no longer present. Note that there is no aggregate income risk in our environment, hence storage would never be used at the first best. In other words, the aggregate asset dynamics in our model are due to incomplete markets generated by the LC friction.⁶

The introduction of public storage has new qualitative implications for the dynamics of consumption predicted by the model when assets are stochastic at the steady state. First, the amnesia property, i.e., that whenever an agent's participation constraint (PC) binds the consumption allocation depends only on his current income (Kocherlakota, 1996), does not hold. Second, the persistence property of the basic model, i.e., that for 'small' income changes consumption is constant, does not hold either. There is a common intuition behind these results: the past history of shocks affects current consumptions through aggregate assets.

We also show that constrained-efficient allocations can be decentralised as competitive equilibria with endogenous borrowing constraints (Alvarez and Jermann, 2000) and a competitive financial intermediation sector which runs the storage technology (Ábrahám and Cárceles-Poveda, 2006). In this environment, equilibrium asset prices take into account the

⁶As we discuss in Section 5, allowing for stochastic aggregate income is without conceptual difficulty and most of our characterisation would go through. In terms of computation, it would simply add one more state variable, as one would have to also condition on aggregate income either directly or through the second agent's income realisation. We assume a constant aggregate income to isolate the effects of partial risk sharing on savings.

externality of aggregate storage on default incentives. In this sense, our paper provides a joint theory of endogenous borrowing constraints and an endogenously growing (and shrinking) asset/Lucas tree in equilibrium.

We then consider hidden storage as well. We show that for a wide range of storage returns, from possibly negative to the discount rate, agents would store privately without public storage,⁷ but they no longer have an incentive to store in a hidden way at the constrained-efficient allocation in our model with public storage.⁸ In other words, with optimal public asset accumulation, the social planner preempts the agents' private storage incentives. This is true because the planner has more incentive to store than the agents. First, the planner stores for the agents, because she inherits their preference for consumption smoothing. Second, storage by the planner makes it easier to satisfy agents' PCs in the future. In other words, the planner internalises the positive externality generated by accumulated assets on future risk sharing.

This result means that the characteristics of constrained-efficient allocations in a model with both public and private storage and a model with only public storage are the same. This result also implies that agents' Euler inequalities are always satisfied in our model with LC and public storage for reasonable rates of return on storage. The Euler inequality cannot be rejected in micro data from developed economies, once labour supply decisions and demographics are appropriately accounted for (Attanasio, 1999). Therefore, we bring LC models in line with this third observation about consumption dynamics as well.

In a private-information environment with full commitment, Cole and Kocherlakota (2001) show that public storage is never used and agents' private saving incentives are binding in equilibrium, eliminating any risk-sharing opportunity beyond self-insurance.⁹ When the deep friction is LC as opposed to private information, the results are very different: first, public storage is used in equilibrium, and second, private storage incentives do not bind. The main difference between the two environments is that in our environment more public storage helps to reduce the underlying LC friction, while with private information public asset accumulation would make incentive provision for truthful revelation more costly.

We also study what the overall effects of access to storage are on welfare. These crucially depend on the return on storage. The availability of private storage increases the value of

⁷Alvarez and Jermann (2000) show that in the basic model agents' Euler equation holds if the interest rate is the equilibrium return on a risk-free bond implied by the constrained-efficient allocation. Our result holds also for any storage return above that equilibrium rate but below the discount rate.

⁸Note that this result does not hinge on how agents' outside option is specified precisely: they may or may not be allowed to store privately in autarky, and they may or may not face additional punishments for defaulting.

⁹See also Allen (1985) and Ábrahám, Koehne, and Pavoni (2011).

autarky, which reduces welfare, while accumulated public assets improve steady-state welfare, both by decreasing consumption dispersion and increasing available resources. When the return on storage is sufficiently high, there are welfare gains at the steady state, because the economy gets close to perfect risk sharing, and aggregate consumption is higher than in the basic model. When the return on storage is lower, the negative effect of a better outside option dominates the positive effect of public assets on welfare. In the short run, public asset accumulation also has costs in terms of foregone consumption. Obviously, compared to private storage only, public asset accumulation always improves welfare. To see whether access to both private and public storage improves welfare taking into account the transition from the moment storage becomes available, we propose an algorithm to solve the model numerically. It turns out that when initial risk sharing is low and the return on storage is high, the steady-state welfare gains dominate the short-term losses, i.e., access to public storage is welfare-improving even if it is accompanied by private storage opportunities.

Finally, we perform a simple regression analysis of consumption dynamics to compare our model and the basic model to the data. We are interested in whether our model is able to better capture how consumption reacts to current and past idiosyncratic income shocks. That is, we study whether the failure of the amnesia and persistence properties happens in an empirically relevant way. We compare how past incomes matter in (i) simulated data from the basic model, (ii) simulated data from our model with both public and private storage, and (iii) households survey data from three Indian villages collected by the International Crop Research Institute for the Semi-Arid Tropics (ICRISAT). We find that, controlling for past consumptions and current incomes, past incomes are insignificant in explaining consumptions in the basic model, and they have a significant negative effect in both our model and the data. Hence, our extension has the potential to improve the LC framework’s capacity to explain consumption dynamics.

The rest of the paper is structured as follows. Section 2 introduces and characterises our model with public storage. Section 3 shows that agents’ private storage incentives are eliminated under optimal public asset accumulation. Section 4 presents some computed examples and our regression analysis of consumption dynamics. Section 5 concludes with a summary and provides a broader perspective on our results. Appendices contain (A) our longer proofs, (B) a description of our numerical algorithm to solve the model, (C) our approach to verify the validity of the first-order-conditions approach in our model with both public and private storage, and (D) a brief description of the ICRISAT data we use.

2 The model with public storage

We consider an endowment economy with two agents, $i = \{1, 2\}$, who are infinitely lived and risk averse. All agents are ex-ante identical in the sense that they have the same preferences and are endowed with the same exogenous random endowment process.¹⁰ Let $u(\cdot)$ denote the utility function. We assume that it is characterised by harmonic absolute risk aversion (HARA), i.e., $u(c) = \xi (b + c/\sigma)^{1-\sigma}$, where $c \geq 0$ is consumption, and $b \geq 0$ and $\sigma > 0$.¹¹ We set $\xi = \sigma^{-\sigma}$ and $a = b\sigma^{-\sigma}$ to have $u'(c) = (a + c)^{-\sigma}$. Note that HARA utility functions satisfy prudence, i.e., $u'''(\cdot) > 0$. We further assume that inverse marginal, $1/u'(\cdot)$, is convex, that is, $\sigma \geq 1$. Some of our results, in particular, those relating to the long run, hold under weaker assumptions. The common discount factor is denoted by β .

Let s_t denote the state of the world realised at time t and s^t the history of endowment realisations, that is, $s^t = (s_1, s_2, \dots, s_t)$. Given s_t , agent 1 has income $y(s_t) > 0$, while agent 2 has income equal to $Y - y(s_t) > 0$, where Y is the aggregate endowment. Note that there is no aggregate risk in the sense that the aggregate endowment is constant.¹² However, the distribution of income varies over time. We further assume that income has a discrete support with J elements, that is, $y(s_t) \in \{y^1, \dots, y^j, \dots, y^J\}$ with $y^j < y^{j+1}$, and is independently and identically distributed (i.i.d.) over time, that is, $\Pr(y(s_t) = y^j) = \Pr(y^j) = \pi^j, \forall t$. The assumptions that there are two types of agents and no aggregate risk impose some symmetry on both the income realisations and the probabilities. In particular, $y^j = Y - y^{J-j+1}$ and $\pi^j = \pi^{J-j+1}$. The i.i.d. assumption can be relaxed, we only need weak positive dependence, i.e., that expected future income is weakly increasing in current income.¹³

Suppose that risk sharing is limited by two-sided lack of commitment to risk sharing contracts, i.e., insurance transfers have to be voluntary, or, self-enforcing, as in [Thomas and Worrall \(1988\)](#), [Kocherlakota \(1996\)](#), and others. Each agent may decide at any time and state to default and revert to autarky. This means that only those risk-sharing contracts are sustainable which provide a lifetime utility at least as great as autarky after any history of endowment realisations for each agent. We assume that the punishment for deviation is exclusion from risk-sharing arrangements in the future. This is the most severe subgame-perfect punishment in this context. In other words, it is an optimal penal code in the sense of [Abreu \(1988\)](#) ([Kocherlakota, 1996](#)). Note that so far our setting is identical to that of

¹⁰When we consider the decentralisation of our economy in [Section 2.5](#), we will assume that each agent represents a continuum of ex-ante and ex-post identical agents.

¹¹Note that relative risk aversion is constant for $b = 0$, and we get exponential utility as σ approaches infinity.

¹²In [Section 5](#) we discuss the case with aggregate risk.

¹³In [Section 4.3](#) we allow income to follow a Markov process.

Kocherlakota (1996).

We introduce a storage technology, which makes it possible to transfer resources from today to tomorrow. Assets stored earn a net return r , with $-1 \leq r \leq 1/\beta - 1$. Note that if $r = -1$ we are back to the basic LC model of Kocherlakota (1996). In this section we only allow for public storage, to which defaulting agents do not have access (as in Krueger and Perri, 2006). In Section 3 we also allow agents to store privately both in autarky and along the equilibrium path in a hidden (unobservable and/or non-contractible) way.

The constrained-efficient risk-sharing contract is the solution to the following optimisation problem:

$$\max_{c_i(s^t)} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t)), \quad (1)$$

where λ_i is the (initial) Pareto-weight of agent i , $\Pr(s^t)$ is the probability of history s^t occurring, and $c_i(s^t)$ is consumption by agent i at time t when history s^t has occurred; subject to the resource constraints,

$$\sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1+r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \quad \forall s^t, \quad (2)$$

where $B(s^t)$ denotes public storage when history s^t has occurred, with $B(s^0)$ given; and the participation constraints (PCs),

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq U_i^{au}(s_t), \quad \forall s^t, \forall i, \quad (3)$$

where $\Pr(s^r | s^t)$ is the conditional probability of history s^r occurring given that history s^t occurred up to time t , and $U_i^{au}(s_t)$ is the expected lifetime utility of agent i when in autarky if state s_t has occurred today. In mathematical terms,

$$U_1^{au}(s_t) = u(y(s_t)) + \frac{\beta}{1-\beta} \sum_{j=1}^J \pi^j u(y^j) \quad (4)$$

and

$$U_2^{au}(s_t) = u(Y - y(s_t)) + \frac{\beta}{1-\beta} \sum_{j=1}^J \pi^j u(y^j).$$

The above definition of autarky assumes that agents cannot use the storage technology in autarky. Note, however, that the qualitative results remain the same under different outside options as long as the strict monotonicity of the autarky value in current income is maintained. For example, agents could save in autarky (as in Krueger and Perri, 2006, and in Section 3), or they might endure additional punishments from the community for defaulting (as in Ligon, Thomas, and Worrall, 2002).

2.1 Characterisation preliminaries

We focus on the characteristics of constrained-efficient allocations. Our characterisation is based on the recursive-Lagrangian approach of [Marcet and Marimon \(1998/2017\)](#). However, the same results can be obtained using the promised-utility approach ([Abreu, Pearce, and Stacchetti, 1990](#)).

Let $\beta^t \Pr(s^t) \mu_i(s^t)$ denote the Lagrange multiplier on the PC, (3), and let $\beta^t \Pr(s^t) \gamma(s^t)$ be the Lagrange multiplier on the resource constraint, (2), when history s^t has occurred. The Lagrangian is

$$\begin{aligned} \mathcal{L} = & \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \left\{ \sum_{i=1}^2 \left[\lambda_i u(c_i(s^t)) \right. \right. \\ & \left. \left. + \mu_i(s^t) \left(\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) - U_i^{au}(s_t) \right) \right] \right. \\ & \left. + \gamma(s^t) \left(\sum_{i=1}^2 (y_i(s_t) - c_i(s^t)) + (1+r)B(s^{t-1}) - B(s^t) \right) \right\}, \end{aligned}$$

with $B(s^t) \geq 0$. Note that our problem is convex, because the objective function is strictly concave, the resource constraint is linear, and it has been shown that the PCs define a convex set in models with LC and capital accumulation as long as autarky utility does not depend on the capital stock, B here, see [Sigouin \(2003\)](#).¹⁴ Therefore, the first-order conditions we derive below are both necessary and sufficient, and the solution is unique. Note also that existence is guaranteed as well, because the constraint set is compact. This requires B to be bounded, which we establish below.

Using the ideas of [Marcet and Marimon \(1998/2017\)](#), we can write the Lagrangian in the form

$$\begin{aligned} \mathcal{L} = & \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \left\{ \sum_{i=1}^2 \left[M_i(s^t) u(c_i(s^t)) - \mu_i(s^t) U_i^{au}(s_t) \right] \right. \\ & \left. + \gamma(s^t) \left(\sum_{i=1}^2 (y_i(s_t) - c_i(s^t)) + (1+r)B(s^{t-1}) - B(s^t) \right) \right\}, \end{aligned}$$

where $M_i(s^t) = M_i(s^{t-1}) + \mu_i(s^t)$ and $M_i(s^0) = \lambda_i$. The first-order condition with respect to agent i 's consumption when history s^t has occurred is

$$\frac{\partial \mathcal{L}}{\partial c_i(s^t)} = M_i(s^t) u'(c_i(s^t)) - \gamma(s^t) = 0. \quad (5)$$

¹⁴See also [Thomas and Worrall \(1994\)](#) and [Ábrahám and Cárceles-Poveda \(2006\)](#).

Combining such first-order conditions for agent 1 and agent 2, we have

$$x(s^t) \equiv \frac{M_1(s^t)}{M_2(s^t)} = \frac{u'(c_2(s^t))}{u'(c_1(s^t))}. \quad (6)$$

Here $x(s^t)$ is the current Pareto weight of agent 1 relative to agent 2.¹⁵ Defining $v_i(s^t) \equiv \mu_i(s^t)/M_i(s^t)$, and using the definitions of $x(s^t)$ and $M_i(s^t)$, we can obtain the law of motion

$$x(s^t) = x(s^{t-1}) \frac{1 - v_2(s^t)}{1 - v_1(s^t)}. \quad (7)$$

Note that $0 \leq v_i \leq 1, \forall i$.

The planner's Euler inequality, i.e., the optimality condition for $B(s^t)$, is

$$\gamma(s^t) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \gamma(s^{t+1}).$$

Using (5), (6), and (7), the planner's Euler becomes

$$u'(c_i(s^t)) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \frac{u'(c_i(s^{t+1}))}{1 - v_i(s^{t+1})}, \quad (8)$$

where $0 \leq 1 - v_i(s^{t+1}) \leq 1, \forall s^{t+1}, \forall i$. Given the definition of $v_i(s^{t+1})$ and equation (7), it is easy to see that (8) represents exactly the same mathematical relationship for both agents.

Equation (8) determines the choice of public storage, $B(s^t)$. It is clear that, first, the higher the return on storage is, the more incentive the planner has to store. Second, note that whenever $v_i(s^{t+1}) = 0, \forall s^{t+1}$, (8) is identical with agent i 's standard Euler equation. This implies that whenever risk sharing is not perfect, that is, $c_i(s^{t+1})$ varies over s^{t+1} for a given s^t , the (utilitarian) planner has a precautionary motive for storage inherited from the agents, a typical motive for saving in models with (endogenously or exogenously) incomplete markets. Third, the new term compared to standard models is $1/(1 - v_i(s^{t+1})) \geq 1$. This term is strictly greater than 1 for states when agent i 's PC is binding. Hence, binding future PCs amplify the return on storage. This is because higher public storage makes the PCs looser in the future by reducing the relative attractiveness of default. The planner internalises this effect when choosing the level of public storage.

Next, we introduce some useful notation and show more precisely the recursive formulation of our problem, using the relative Pareto weight, x , as a co-state variable. This recursive formulation is going to be the basis for both the analytical characterisation and the numerical

¹⁵To reinforce this interpretation, notice that if no PC binds in history s^t for either agent, i.e., $\mu_1(s^\tau) = \mu_2(s^\tau) = 0$ for all subhistories $s^\tau \subseteq s^t$, then $x(s^t) = \lambda_1/\lambda_2$, the initial relative Pareto weight of agent 1.

solution procedure. Let $\mathcal{C}()$ and y denote, respectively, the consumption function and current income of agent 1, $\mathcal{V}()$ his value function, $v_i()$ the function determining the normalised Lagrange multiplier on agent i 's PC, $i \in \{1, 2\}$, $x'()$ the function determining the current Pareto weight, and $\mathcal{B}'()$ the function determining public storage. The following system is recursive with $X = (y, B, x)$ as state variables:

$$x'(X) = \frac{u'(Y + (1+r)B - \mathcal{B}'(X) - \mathcal{C}(X))}{u'(\mathcal{C}(X))} \quad (9)$$

$$x'(X) = x \frac{1 - v_2(X)}{1 - v_1(X)} \quad (10)$$

$$u'(\mathcal{C}(X)) \geq \beta(1+r) \sum_{y'} \Pr(y') \frac{u'(\mathcal{C}(X'))}{1 - v_1(X')} \quad (11)$$

$$u(\mathcal{C}(X)) + \beta \sum_{y'} \Pr(y') \mathcal{V}(X') \geq U^{au}(y) \quad (12)$$

$$u(Y + (1+r)B - \mathcal{B}'(X) - \mathcal{C}(X)) + \beta \sum_{y'} \Pr(y') \mathcal{V}(Y - y', \mathcal{B}'(X), 1/x'(X)) \geq U^{au}(Y - y) \quad (13)$$

$$\mathcal{B}'(X) \geq 0. \quad (14)$$

The first equation, (9), where we have used the resource constraint to substitute for the consumption of agent 2, says that the ratio of marginal utilities between the two agents has to be equal to the current relative Pareto weight. Equation (10) is the law of motion of the co-state variable. Equation (11) is the social planner's Euler inequality, which we have derived above. Equations (12) and (13) are the PCs of agent 1 and agent 2, respectively. Finally, equation (14) makes sure that public storage is never negative.

Given the recursive formulation above, and noting that the outside option $U^{au}()$ is monotone in current income and takes a finite number of values, the solution can be characterised by a set of state-dependent intervals on the current Pareto weight. This is analogous to the basic model, where public storage is not considered (see [Ljungqvist and Sargent, 2012](#), for a textbook treatment). The key difference is that these optimal intervals on the relative Pareto weight depend not only on current endowment realisations but also on B . The following lemma will be useful for specifying the optimal state-dependent intervals, and hence for characterising the dynamics of our model.

Lemma 1. $\mathcal{C}(\tilde{y}, B, \tilde{x}) = \mathcal{C}(\hat{y}, B, \hat{x})$, $\mathcal{B}'(\tilde{y}, B, \tilde{x}) = \mathcal{B}'(\hat{y}, B, \hat{x})$, and $\mathcal{V}(\tilde{y}, B, \tilde{x}) = \mathcal{V}(\hat{y}, B, \hat{x})$ for all $(\tilde{y}, \tilde{x}), (\hat{y}, \hat{x})$ such that $x'(\tilde{y}, B, \tilde{x}) = x'(\hat{y}, B, \hat{x})$. That is, for determining consumptions, public storage, and agents' expected lifetime utilities, the current relative Pareto weight, x' , is a sufficient statistic for the current income state, y , and the inherited relative Pareto weight, x .

Proof. Given x' and B , equations (9) and (11), either as equality or implying zero public storage, give consumption and public storage as a function of x' and B only. Note that the inherited relative Pareto weight, x , and the current income state, y , do not appear in these equations separately. Then, the left-hand side of (12) gives lifetime utility, where again x and y do not appear separately, and we have proven above that $\mathcal{B}'()$ and $\mathcal{C}()$ are functions of x' and B only. \square

Lemma 1 implies that we can express consumptions, public storage, and agents' lifetime utilities in terms of accumulated assets and the current relative Pareto weight. That is, we can write consumption by agent 1, public assets, and the value function as $c(B, x')$, $B'(B, x')$, and $V(B, x')$, respectively.

The following conditions define the lower and upper bounds of the optimal intervals in state y^j as a function of B :

$$V(B, \underline{x}^j(B)) = U^{au}(y^j) \quad \text{and} \quad V\left(B, \frac{1}{\bar{x}^j(B)}\right) = U^{au}(Y - y^j). \quad (15)$$

Hence, given the inherited relative Pareto weight, x , and accumulated assets, B , the updating rule is

$$x' = \begin{cases} \bar{x}^j(B) & \text{if } x > \bar{x}^j(B) \\ x & \text{if } x \in [\underline{x}^j(B), \bar{x}^j(B)] \\ \underline{x}^j(B) & \text{if } x < \underline{x}^j(B) \end{cases} .$$

The ratio of marginal utilities is kept constant whenever this does not violate the PC of either agent. When the PC binds for agent 1 (agent 2), the relative Pareto weight moves to the lower (upper) limit of the optimal interval, just making sure that this agent is indifferent between staying and defaulting. Given that the value of autarky is strictly increasing in current income, and the value function is strictly increasing in the current Pareto weight, $\bar{x}^j(B) > \bar{x}^{j-1}(B)$ and $\underline{x}^j(B) > \underline{x}^{j-1}(B)$ for all $J \geq j > 1$ and B .

2.2 The dynamics of public assets and the consumption distribution

In the basic model aggregate consumption is constant, and hence it is obvious that an agent's consumption increases with his own Pareto weight. With public storage, aggregate consumption varies as $(1+r)B - B'(x', B)$ varies over time, which depends on x' . Hence, an increase in the current relative Pareto weight, in principle, may imply a sufficiently large decrease in aggregate consumption so that agent 1's consumption decreases. For now we assume this intuitive property that c increases in x' . Formally:

Property 1. *If $\tilde{x}' > \hat{x}'$ then $c(B, \tilde{x}') > c(B, \hat{x}')$, $\forall B$. That is, consumption by agent 1 is strictly increasing in his current relative Pareto weight.*

Next we continue our characterisation assuming that Property 1 holds. In particular, we prove Claim 1 and Proposition 1 using this property. However, to prove Proposition 1, we only need Property 1 to hold in the case where r is such that assets are constant at the steady state. Afterwards, using the results in Claim 1 and Proposition 1 and the uniqueness of the solution established above, we prove that Property 1 indeed holds when assets converge to a constant level in the long run, see Lemma 2. Further, we verify numerically that this property holds also in the case when public assets are stochastic in the long run.

We can describe the dynamics of the model with similar optimal intervals and updating rule on consumption as on the relative Pareto weight, as for the basic model, assuming that Property 1 holds. Using (9) we can implicitly define the limits of the optimal intervals on consumption as

$$\bar{c}^j(B) : \bar{x}^j(B) = \frac{u'(Y + (1+r)B - B'(B, \bar{x}^j(B)) - \bar{c}^j(B))}{u'(\bar{c}^j(B))}$$

and $\underline{c}^j(B) : \underline{x}^j(B) = \frac{u'(Y + (1+r)B - B'(B, \underline{x}^j(B)) - \underline{c}^j(B))}{u'(\underline{c}^j(B))}.$

Symmetry implies that $\bar{c}^j(B) = Y + (1+r)B - B'(B, \bar{x}^j(B)) - \underline{c}^{J-j+1}(B)$. Note further that the monotonicity of $\bar{x}^j(B)$ and $\underline{x}^j(B)$ in income and Property 1 imply that $\bar{c}^j(B)$ and $\underline{c}^j(B)$ are also monotone in income.

These consumption intervals are key to understanding the steady-state distribution of storage as they are the key determinants of risk sharing in the long run. Given the utility function, the income process, r , and B , the intervals for different states may or may not overlap depending on the discount factor, β . The higher β is, the wider these intervals are. By a standard folk theorem (Kimball, 1988), for β sufficiently high, all intervals overlap. That is, $\bar{c}^1(B) \geq \underline{c}^J(B)$, hence perfect risk sharing is implementable. At the other extreme, when β is sufficiently low, agents stay in autarky ($\bar{c}^j(B) = \underline{c}^j(B)$, $\forall j$). Similarly, given the income process, r , and β , as public assets are accumulated (or decumulated) the optimal intervals change. In particular, the intervals are wider when B is higher.¹⁶

When partial insurance occurs, it is useful to distinguish two possible scenarios. To fix ideas, suppose that B is constant at B^* . For higher levels of β and/or B^* , $\bar{c}^j(B^*) \geq \underline{c}^J(B^*) > \bar{c}^1(B^*) \geq \underline{c}^j(B^*)$ for all $1 < j < J$. In this case, it is easy to see that,

¹⁶This is easy to see from (15). Take the first equality. The right-hand side is independent of B , and the value function on the left-hand side is increasing in both its arguments (available resources and own relative Pareto weight), hence as B increases $\underline{x}^j(B)$ and hence $\underline{c}^j(B)$, the lower limit of the optimal interval in state y^j , must decrease. Similarly, from the second inequality in (15), $1/\bar{x}^j(B)$ must decrease, hence $\bar{x}^j(B)$ and $\bar{c}^j(B)$, the upper limit, must increase. Moreover, $\bar{x}^j(B)$ is strictly increasing and $\underline{x}^j(B)$ is strictly decreasing in B for all j , as long as the length of the j -interval is not zero.

in the long run, consumption will only take the values $\underline{c}^J(B^*)$ and $\bar{c}^1(B^*)$. This implies that the consumption distribution is time-invariant in the long run in the sense that the mass of people consuming either of these values is constant. This does not mean that individual consumption will be constant. Instead, it will fluctuate between these values, and change whenever a PC is binding, i.e., the agent who consumed $\bar{c}^1(B^*)$ yesterday gets the highest income, y^J , today, and hence his consumption jumps to $\underline{c}^J(B^*)$. For lower levels of β and/or B^* , we have that $\bar{c}^j(B^*) < \underline{c}^J(B^*)$ and $\underline{c}^j(B^*) > \bar{c}^1(B^*)$ for some j . In turn, consumption will take more than two values in the long run in this case, and consequently the consumption distribution will also fluctuate over time. Note that this characterisation also holds for the basic limited-commitment model, where $B^* = 0$.

The next claim provides a key property of the public storage decision rule and characterises the short-run dynamics of assets. It shows how public storage varies with the consumption and income distribution.

Claim 1. *Under Property 1, $B'(B, x')$ is increasing in x' for $x' \geq 1$ and $B'(B, x') > 0$. That is, the higher cross-sectional consumption inequality is, the higher public asset accumulation is. Further, consider $y^j > y^k \geq Y/2$ (or $y^j < y^k \leq Y/2$). Then, $\mathcal{B}'(y^j, B, x) \geq \mathcal{B}'(y^k, B, x)$, $\forall(B, x)$. That is, aggregate asset accumulation is increasing with cross-sectional income inequality.*

Proof. In Appendix A.

The intuition for Claim 1 is coming from two related observations. Higher inequality in the current period implies higher expected consumption inequality/risk next period. Under convex inverse marginal utility, the planner has a higher precautionary motive for saving whenever she faces more risk tomorrow.¹⁷

We are now ready to characterise the steady-state behaviour of public assets and the consumption distribution.¹⁸

Proposition 1. *Assume that β is such that the consumption distribution is time-varying without public storage.*

¹⁷Note that with log utility B' is weakly increasing in $x' \geq 1$, i.e., in cross-sectional consumption inequality, since $1/u'$ is linear in this case, while for CRRA utility functions with a coefficient of relative risk aversion strictly greater than 1, the empirically more plausible range, $1/u'$ is strictly convex, hence B' is strictly increasing in $x' \geq 1$.

¹⁸Note that Proposition 1 below implies that the stochasticity of the steady state crucially depends on r . In particular, in case (i) the consumption distribution and individual consumptions, but not assets, are stochastic; in case(ii) all these variables remain stochastic; in case (iii) only individual consumptions are stochastic; and finally, in case (iv) all variables converge to a constant.

- (i) *There exists r_1 such that for all $r \in [-1, r_1]$, public storage is never used in the long run.*
- (ii) *There exists a strictly positive $r_2 > r_1$ such that for all $r \in (r_1, r_2)$, B remains stochastic but bounded, and the consumption distribution is time-varying in the long run.*
- (iii) *For all $r \in [r_2, 1/\beta - 1)$, B converges almost surely to a strictly positive constant, B^* , which is independent of the initial level of assets, and where the consumption distribution is time-invariant, but perfect risk sharing is not achieved.*
- (iv) *Whenever $r = 1/\beta - 1$, B converges almost surely to a strictly positive constant and perfect risk sharing is self-enforcing.*

If β is such that the consumption distribution is time-invariant without public storage, then $r_1 = r_2$, hence only (i), (iii), and (iv) occur.

Proof. In Appendix A.

The intuition behind Proposition 1 is that the social planner trades off two effects of increasing public storage: it is costly as long as $\beta(1+r) < 1$, but less so the higher r is, and it is beneficial because it reduces consumption dispersion in the future. The level of public assets chosen just balances these two opposing forces, the relative strength of which naturally depends on the return on storage, r . When the cross-sectional consumption distribution is time-varying (case (ii)), the relative strength of the two forces determining asset accumulation changes over time. This implies that assets cannot settle at a constant level in this case.

When the return on storage is sufficiently high (case (iii)), assets are accumulated so that PCs are only binding for agents with the highest income in the long run, and the consumption distribution becomes time-invariant. In this case, there is a unique constant level of assets, B^* , which exactly balances the trade-off between impatience and the risk-sharing gains of public storage. Finally, in the limiting case of $\beta(1+r) = 1$ (case (iv)), the planner does not face a trade-off between improving risk sharing and using an inefficient intertemporal technology, hence assets are accumulated until the level where full insurance is enforceable.^{19,20}

We now show that Property 1 holds in the case where assets converge to a constant level in the long run. We first show that Claim 1 and Proposition 1 imply that Property 1 holds.

¹⁹If the initial level of assets is high enough so that perfect risk sharing is self-enforcing, then assets stay at their initial level. Therefore, unlike in case (iii), the steady-state level of assets is not unique.

²⁰Note that similar results can be obtained in a model of one-sided LC of an open economy with a continuum of agents, competitive financial intermediaries, no punishment for default for agents, and if principals/intermediaries are at least as patient as agents. In that environment, Krueger and Uhlig (2006) show that whenever $\beta(1+r) = 1$ ($\beta(1+r) < 1$ but not ‘too far’ from 1), the steady-state outcome is full (partial) risk sharing, and the economy is accumulating assets against the intermediaries (the rest of the world). See also Ljungqvist and Sargent (2012).

Then, given the uniqueness of the solution, this implies that the solution is characterised by Property 1 along the transition as well when r is such that assets are constant in the long run.

Lemma 2. *If $\tilde{x}' > \hat{x}'$ then $c(B, \tilde{x}') > c(B, \hat{x}')$, $\forall B$, as long as assets converge to a constant level in the long run. That is, consumption by agent 1 is strictly increasing in his current relative Pareto weight.*

Proof. In Appendix A.

The intuition for Lemma 2 is the following. As a response to increasing inequality, it cannot be optimal to increase public storage so much that both agents have lower consumption. That would contradict the optimal intertemporal smoothing behaviour of the planner.²¹

We illustrate the dynamics of assets in the case where they are stochastic in the long run in Figure 1.²² For simplicity, we consider three income states, indexed by the income of agent 1, $y^l < y^m < y^h$. This means that there are two types of states: two with high income and consumption inequality (states y^h and y^l) and one with low income and consumption inequality (state y^m). The solid line represents $B'(B, \underline{x}^h(B))$, i.e., public storage in state y^h (or y^l) when the relevant PC is binding. Similarly, the dot-dashed line represents $B'(B, \bar{x}^m(B))$, i.e., storage in state y^m when the relevant PC is binding. Starting from B_0 , if state y^m occurs repeatedly, assets converge to the lower limit of their stationary distribution, denoted \underline{B} . The relevant PC is always binding along this path, because assets keep decreasing.

The dashed line represents the scenario where state y^h (or state y^l) occurs when assets are at the lower limit of the stationary distribution, \underline{B} , and then the same state occurs repeatedly. This is when assets approach the upper limit of their stationary distribution, denoted \bar{B} . The relevant PC is not binding from the period after the switch to y^h , therefore public storage given assets is described by the function $B'(B, \underline{x}^h(\underline{B}))$.

Under Property 1 we can also characterise analytically the limits of the long-run stationary distribution of assets, for any number of income states, as follows.

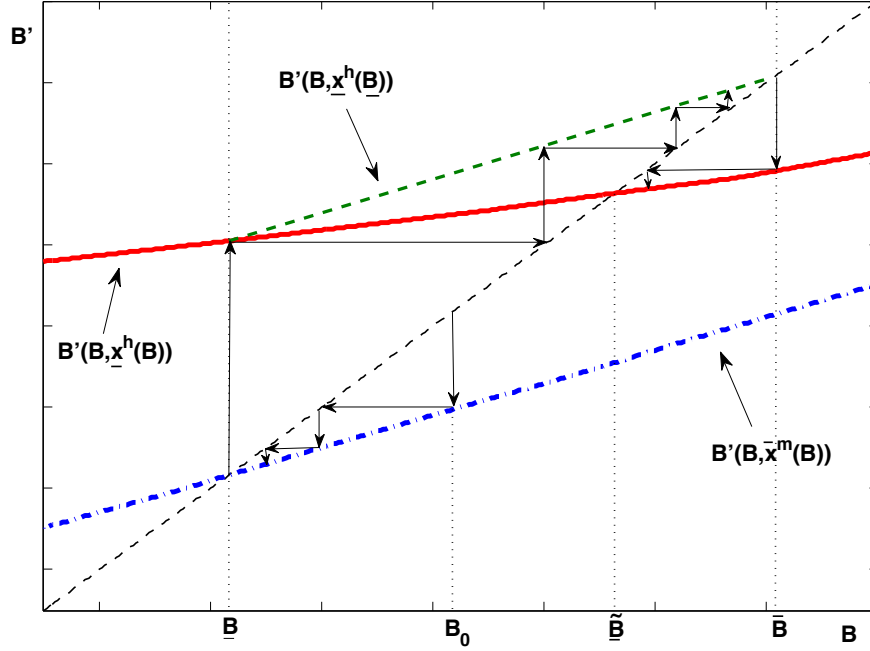
Claim 2. *Under Property 1, the lower limit of the stationary distribution of public assets, \underline{B} , is either strictly positive and is implicitly given by*

$$u'(\bar{c}^m(\underline{B})) = \beta(1+r) \sum_{j=1}^J \pi^j \frac{u'(C(y^j, \underline{B}, \bar{x}^m(\underline{B})))}{1 - v_1(y^j, \underline{B}, \bar{x}^m(\underline{B}))}, \quad (16)$$

²¹Note that the analytical proof can be extended to some values of r which imply that assets are stochastic in the long run, in particular, values which are close to the threshold above which assets are constant in the long run or to the threshold below which zero public storage is optimal.

²²We have generated the figure numerically and verified that Property 1 holds.

Figure 1: Asset dynamics when assets are stochastic at the steady state



or is zero and (16) holds as strict inequality. The upper limit of the stationary distribution of public assets, \bar{B} , is implicitly given by

$$u'(\mathcal{C}(y^J, \bar{B}, \underline{x}^J(\underline{B}))) = \beta(1+r) \sum_{j=1}^J \pi^j u'(\mathcal{C}(y^j, \bar{B}, \underline{x}^j(\underline{B}))). \quad (17)$$

Proof. In Appendix A.

Finally, assume, without loss of generality, that state y^1 occurred many times while approaching \bar{B} , and suppose that state y^J occurs when assets are (close to) \bar{B} . In this case, $x' = \underline{x}^J(\bar{B}) < \underline{x}^J(\underline{B})$, and assets decrease. They then converge to a level \tilde{B} from above with the relevant PC binding along this path. The same happens whenever $B > \tilde{B}$ when we switch to state y^J (or y^1). \tilde{B} is implicitly given by

$$u'(\underline{c}^J(\tilde{B})) = \beta(1+r) \sum_{j=1}^J \pi^j u'(\mathcal{C}(y^j, \tilde{B}, \underline{x}^h(\tilde{B}))).$$

2.3 The dynamics of individual consumptions

Having characterised assets, we now turn to the dynamics of consumption. One key property of the basic model is that whenever either agent's PC binds ($v_1(X) > 0$ or $v_2(X) > 0$), the resulting allocation is independent of the preceding history. In our formulation, this implies

that x' is only a function of y and the identity of the agent with a binding PC. This is often called the amnesia property (Kocherlakota, 1996). Typically data do not support this pattern, see Broer (2013) for the United States and Kinnan (2017) for Thai villages. Allowing for public storage helps to bring the model closer to the data in this respect.

Proposition 2. *The amnesia property does not hold when public assets are stochastic at the steady state.*

Proof. x' and hence current consumption depend on both current income and assets, B , when a PC binds. This implies that the past history of income realisations affects current consumptions through B . \square

Another property of the basic model is that whenever neither PC binds ($v_1(X) = v_2(X) = 0$), the consumption allocation is constant and hence exhibits an extreme form of persistence.²³ It is again not easy to find evidence for this pattern in the data, see Broer (2013). In our model, even if the relative Pareto weight does not change, (9) does not imply that individual consumptions will be the same next period as in the current period.

Proposition 3. *The persistence property does not hold generically when public assets are stochastic at the steady state.*

Proof. Even though $x' = x$, $(1+r)B - \mathcal{B}'(X)$ is generically not equal to $(1+r)B' - \mathcal{B}''(X')$ when assets are stochastic at the steady state.²⁴ \square

The last two propositions imply that in our model the dynamics of consumption are richer than in the basic model in a qualitative sense. We provide reduced-form regression evidence in Section 4.3 below that history dependence occurs in a similar way in our model as in data from village economies. It should be clear from this discussion that the failure of these properties is due to changes of an endogenous aggregate state variable, public assets. However, typically when these properties are tested within households (see e.g. Lise and Yamada, 2015), household assets are not controlled for. Similarly, when the amnesia property is tested in village economies, the non-linear and asymmetric effects of changes in public storage are not taken into account, partly due to a lack of data. In this sense, the above results also provide inputs for the development of more theory-consistent empirical tests for these key properties of LC models.

²³This can be seen easily: (10) gives $x' = x$, and the consumption allocation is only a function of x' with constant aggregate income. This implies that for ‘small’ income changes which do not trigger a PC to bind, we do not see any change in individual consumptions.

²⁴The only exceptions are asset levels \underline{B} , \tilde{B} , and \bar{B} in Figure 1 with the appropriate income states occurring. However, the probability that assets settle at these points in the stationary distribution is zero.

2.4 Welfare

It is clear that access to public storage cannot reduce welfare, because zero assets can always be chosen. Along the same lines, if public storage is strictly positive for at least the most unequal income state, then welfare strictly improves. Proposition 1 implies that this is the case whenever the basic model does not display perfect risk sharing and the return on storage is higher than $r_1 < 1/\beta - 1$.

2.5 Decentralisation

Ábrahám and Cárceles-Poveda (2006) show how to decentralise a LC economy with capital accumulation and production. That economy is similar to the current one in one important aspect: agents are excluded from receiving capital income after default. They introduce competitive intermediaries and show that a decentralisation with endogenous debt constraints which are ‘not too tight’, as in Alvarez and Jermann (2000), is possible.²⁵ Public storage can be thought of as a form of capital, B units of which produce $(1 + r)B$ units of output tomorrow and which fully depreciates. Hence, the results above directly imply that a competitive equilibrium corresponding to the constrained-efficient allocation exists.

In particular, households trade Arrow securities subject to endogenous borrowing constraints which prevent default, and the intermediaries also sell these Arrow securities to build up public storage. The key intuition is that equilibrium Arrow-security prices take into account binding future PCs, as these prices are given by the usual pricing kernel. Moreover, agents do not hold any ‘shares’ in public storage, hence their autarky value is not affected. Finally, no arbitrage or perfect competition guarantees that the intermediaries make zero profits in equilibrium. As opposed to Ábrahám and Cárceles-Poveda (2006), capital accumulation constraints are not necessary, because in our model public storage does not affect agents’ outside option.

3 The model with both public and private storage

In this section, we allow agents to use the same storage technology as the social planner in a private (unobservable and/or non-contractible) way. Access to private storage both

²⁵Ábrahám and Cárceles-Poveda (2006) use a neoclassical production function where wages depend on aggregate capital. This implies that there the value of autarky depends on aggregate capital as well. This is also the case in the two-country production economy of Kehoe and Perri (2004). Ábrahám and Cárceles-Poveda (2006) show that if the intermediaries are subject to endogenously determined capital accumulation constraints, then this externality can be taken into account, and the constrained-efficient allocation can be decentralised as a competitive equilibrium. Chien and Lee (2012) achieve the same objective by taxing capital instead of using a capital accumulation constraint.

affects agents' autarky values and enlarges the set of possible actions and deviations. In mathematical terms, allowing for private storage requires adding agents' Euler inequalities as constraints to the problem given by the objective function (1) and the constraints (2) and (3), and modifying the PCs, (3).

The social planner's problem becomes

$$\begin{aligned}
(P1) \quad & \max_{\{c_i(s^t), B(s^t)\}} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t)) \\
& \text{s.t.} \quad \sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1+r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \quad \forall s^t, \\
& \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq \tilde{U}_i^{au}(s_t), \quad \forall s^t, \forall i, \tag{18}
\end{aligned}$$

$$u'(c_i(s^t)) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1} | s^t) u'(c_i(s^{t+1})), \quad \forall s^t, \forall i. \tag{19}$$

The objective function, the resource constraint, and the non-negativity-of-public-storage restriction remain the same as before. The PCs change slightly, since $\tilde{U}_i^{au}(s_t)$ in (18) is the value function of autarky when private storage is allowed, to be defined precisely below. Agents' Euler inequalities, equation (19), guarantee that agents have no incentive to deviate from the proposed allocation by storing privately. Note that we implicitly assume that private storage is zero at the initial period.

A few remarks are in order about this structure before we turn to the characterisation of constrained-efficient allocations. First, agents can store privately in autarky, but they lose access to the benefits of the public asset. This implies that if the current shock realisation is s , corresponding to income y for agent 1 and $Y - y$ for agent 2, then $\tilde{U}_1^{au}(s) = V^{au}(y, 0)$ and $\tilde{U}_2^{au}(s) = V^{au}(Y - y, 0)$, where $V^{au}(z, b)$ is defined as

$$V^{au}(z, b) = \max_{b' \geq 0} \left\{ u(z + (1+r)b - b') + \beta \sum_{j=1}^J \pi^k V^{au}(y^j, b') \right\}, \tag{20}$$

where b denotes private savings. Since $V^{au}(\cdot, 0)$ is increasing, it is obvious that if we replace the autarky value in the model of Section 2 (or in the basic model) with the one defined here, the same characterisation holds. Note that, unlike in [Bulow and Rogoff \(1989\)](#), state-contingent assets are not available in autarky.²⁶

Second, we use a version of the first-order-conditions approach (FOCA) here. That is, these constraints only cover a subset of possible deviations. In particular, we verify that each

²⁶[Bulow and Rogoff \(1989\)](#) find that access to state-contingent 'cash-in-advance contracts' in autarky prevents risk sharing in equilibrium. However, "[t]his conclusion does depend on a sovereign's ability to reproduce any risk-sharing advantages of loan contracts by holding a portfolio of foreign assets" (p. 49).

agent is better off staying in the risk-sharing arrangement rather than defaulting and possibly storing in autarky (constraint (18), see also (20)), and that he has no incentive to store privately given that he does not ever default (constraint (19), agents' consumption-saving optimality condition). It is not obvious whether these constraints are sufficient to guarantee incentive compatibility, because multiple and multi-period deviations are not considered by these constraints.²⁷ In particular, an agent can store privately in the current period to increase his value of autarky in future periods and default in a later period. For now, we assume that these deviations are not profitable given the contract which solves Problem *P1*. We characterise the solution under this assumption. Then, in Appendix C we provide a numerical verification algorithm to show that agents indeed have no incentive to use these more complex deviations, unless the return on the storage technology is in a small neighbourhood of the efficient level.

Third, both the PCs, (18), and the Euler constraints, (19), involve future decision variables. Given these two types of forward-looking constraints, a recursive formulation using either the promised-utility approach of Abreu, Pearce, and Stacchetti (1990) or the recursive-Lagrangian approach of Marcet and Marimon (1998/2017) is difficult. In this paper, we follow a different approach. In particular, we show that the solution of a simplified problem where agents' Euler inequalities are ignored satisfies those Euler constraints. That is, instead of Problem *P1*, we consider the following simpler problem:

$$\begin{aligned}
 (P2) \quad & \max_{\{c_i(s^t), B(s^t)\}} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t)) \\
 & \text{s.t.} \quad \sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1+r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \quad \forall s^t, \\
 & \quad \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq \tilde{U}_i^{au}(s_t), \quad \forall s^t, \forall i.
 \end{aligned}$$

This is the problem we studied in Section 2, except that the autarky value is different. Now, we are ready to state the main result of this section.

Proposition 4. *The solution of the model with hidden storage, *P1*, corresponds to the solution of the simplified problem, *P2*.*

Proof. We prove this proposition by showing that the allocation which solves *P2* satisfies agents' Euler inequalities, (19), the only additional constraints. Note that the planner's

²⁷In fact, Kocherlakota (2004) shows that in an economy with private information and hidden storage the FOCA can be invalid.

Euler, (8), is a necessary condition for optimality for $P2$. It is clear that the right-hand side of (8) is bigger than the right-hand side of (19), for $i = \{1, 2\}$, since $0 \leq v_i(s^{t+1}) \leq 1, \forall s^{t+1}$. Therefore, (8) implies (19). \square

This result implies that the characteristics of the constrained-efficient allocations of Problem $P1$ are the same as those of Problem $P2$, which is the problem we studied in Section 2. Proposition 4 also means that private storage does not matter as long as public asset accumulation is optimal. We have to emphasise, however, that the result that no private storage occurs hinges on the assumption of optimal public asset accumulation with the same return.

The intuition behind this result is that the planner has more incentive to store than the agents. She stores for the agents, because she inherits their preference for consumption smoothing. Thereby she can eliminate agents' incentive to store in a hidden way. Further, comparing (8) and (19) again, it is obvious that the planner has more incentive to store than the agents in all but the most unequal states. In particular, the presence of $1/(1 - v_i(s^{t+1})) > 1$ in the planner's Euler indicates how public asset accumulation helps the planner to relax future PCs, and thereby improve risk sharing, or, make markets more complete. In other words, the planner internalises the positive externality of public asset accumulation on future risk sharing.

Next, we relate the case with both private and public storage to the case with private storage in autarky but without public storage. The following result follows from Proposition 4.

Corollary 1. *The planner stores in equilibrium whenever an agent's Euler inequality is violated at the constrained-efficient allocation of the basic model with no public storage and private storage only in autarky.*

Corollary 1 says that whenever agents have private storage incentives in the basic model, public storage is used in equilibrium. However, this result is only interesting if private storage matters, i.e., agents' Euler inequalities are violated, in the basic model under general conditions. This is what we establish next.

3.1 Does hidden storage matter in the basic model?

In this section, we identify conditions under which agents would store privately at the constrained-efficient solution of the basic model without public storage. We assume that partial insurance occurs at the solution, because otherwise it is trivial that private storage is never used. If agents' Euler inequalities are violated, the solution of the basic model is not robust to deviations when private storage is available.

We first consider the benchmark case where agents have access to an efficient intertemporal technology, i.e., storage earns a return r such that $\beta(1+r) = 1$. Afterwards, we study the general case. We only examine whether agents would use the available hidden intertemporal technology at the constrained-efficient allocation of the basic model.

Lemma 3. *Suppose that partial insurance occurs and the hidden storage technology yields a return r such that $\beta(1+r) = 1$. Then agents' Euler inequalities are violated at the constrained-efficient allocation of the basic model.*

Proof. We show that the Euler inequality is violated at least when an agent receives the highest possible income, y^J , hence his PC is binding. By the characterisation in Section 2.1, it is clear that for all future income levels his consumption will be no greater than his current consumption, i.e., $\mathcal{C}(y^j, 0, \underline{x}^J(0)) \leq \underline{c}^J(0)$. If partial insurance occurs, then it must be that there exists some state y^k where the agent consumes $\mathcal{C}(y^k, 0, \underline{x}^J) < \underline{c}^J(0)$. Then, $u'(\underline{c}^J(0)) < \sum_{y^j} \pi^j u'(\mathcal{C}(y^j, 0, \underline{x}^J(0)))$, that is, the Euler inequality is violated. \square

The following proposition shows that for all economies with partial insurance one can find a threshold return on storage above which agents' storage incentives bind in the basic model.

Proposition 5. *There exists $\tilde{r} < 1/\beta - 1$ such that for all $r > \tilde{r}$ agents' Euler inequalities are violated at the constrained-efficient allocation of the basic model. Further, \tilde{r} is implicitly defined as*

$$u'(\underline{c}^J(0)) = \beta(1 + \tilde{r}) \sum_{y^j} \pi^j u'(\mathcal{C}(y^j, 0, \underline{x}^J(0))). \quad (21)$$

Proof. For \tilde{r} close to -1 , the right-hand side of (21) is close to zero. For $\tilde{r} = 1/\beta - 1$, the right-hand side is greater than the left-hand side by Lemma 3. It is obvious that the right-hand side is continuous and increasing in \tilde{r} . Therefore, there is a unique \tilde{r} that solves equation (21), and agents' Euler inequalities are violated for higher values of r . \square

The intuition behind this result is that whenever partial insurance occurs, the agent enjoying high consumption in the current period faces a weakly decreasing consumption path. Therefore, if a storage technology with sufficiently high return is available, the agent uses it for self-insurance purposes. One can also show that the threshold \tilde{r} in Proposition 5 can be negative.²⁸

Note that this result does not contradict that of Alvarez and Jermann (2000) that agents' Euler equation holds if the interest rate is the equilibrium return on a risk-free bond implied

²⁸In particular, we have shown that agents would use a hidden storage technology with $r = 0$ under non-restrictive conditions. A necessary condition is that the consumption distribution is time-varying at the steady state. The proofs of these results are available upon request.

by the constrained-efficient allocation when public storage is not allowed, denoted r^* . Given that markets are endogenously incomplete, we know that the equilibrium price is such that $\beta(1 + r^*) < 1$. This implies that public storage can only matter if the return on storage is above r^* .

3.2 The dynamics of individual consumptions revisited

We have shown in Section 2.3 that, introducing public storage, we potentially overturn two counterfactual properties of consumption dynamics in the basic model, the amnesia and persistence properties. We can improve on the basic model with respect to a third aspect of the dynamics of consumption. In particular, the Euler inequality cannot be rejected in household survey data from developed economies, once household demographics and labour supply are appropriately accounted for (see [Attanasio, 1999](#), for a comprehensive review of the literature). Empirically, as long as households have access to an intertemporal technology with a return above r^* , our model, unlike the basic model, is in line with this third observation as well.

3.3 Welfare revisited

In Section 2.4 we have argued that access to public storage unambiguously reduces consumption dispersion and improves welfare. It is clear that hidden storage counteracts these benefits of public storage, because it increases the value of agents' outside option, which in itself increases consumption dispersion and reduces welfare. The overall effects of access to both public and private storage are hence ambiguous in general, and depend on the return to storage, r . We first compare welfare at the (possibly stochastic) steady state of our model with both public and private storage and the basic model without storage. Afterwards, we discuss the effects of the transition from the moment when storage becomes available.

In the following proposition we compare consumption dispersion and (equal-weighted) social welfare at the steady state in two economies. In the first economy neither public nor private storage is available, in the second one both are available.

Proposition 6.

- (i) *There exists \tilde{r}_1 such that for all $r \in [-1, \tilde{r}_1]$ storage is not used even in autarky, therefore access to storage leaves consumption dispersion unchanged and is welfare neutral.*
- (ii) *There exists $\tilde{r}_2 > \tilde{r}_1$ such that for all $r \in (\tilde{r}_1, \tilde{r}_2]$ storage is used only in autarky, therefore consumption dispersion increases and welfare deteriorates as a result of access to storage, and strictly so as long as perfect risk sharing is not self-enforcing.*

- (iii) *There exists $\tilde{r}_3 > \tilde{r}_2$ such that for all $r \in (\tilde{r}_2, \tilde{r}_3)$ public storage is (at least sometimes) strictly positive, but access to storage is still welfare reducing and consumption dispersion is higher than in the basic model without storage. Access to storage is welfare neutral at the steady state at the threshold $r = \tilde{r}_3$.*
- (iv) *There exists $\tilde{r}_4 > \tilde{r}_3$ such that for all $r \in (\tilde{r}_3, \tilde{r}_4)$ access to storage is welfare improving at the steady state, but consumption dispersion is still higher than in the basic model. Consumption dispersion is the same at the threshold $r = \tilde{r}_4$.*
- (v) *For all $r \in (\tilde{r}_4, 1/\beta - 1]$ access to storage is welfare improving at the steady state, and consumption dispersion is lower than in the basic model.*

Proof. In Appendix A.

Even when welfare improves at the steady state, accumulating public assets has short-run costs in terms of reduced aggregate consumption. This implies that the total gains (losses) from gaining access to storage are lower (higher) than those we have considered in Proposition 6. However, it is not clear whether access to both private and public storage improves welfare. For this reason, we explore this issue using numerical examples in the next section.

4 Numerical results

In this section we solve for the constrained-efficient allocation in economies with LC and access to public and private storage. As in Section 3, agents are allowed to store privately in autarky. We describe the algorithm we have applied in more detail in Appendix B.

First, we consider the simplest possible setting in order to illustrate the working of the model, focusing on public assets, consumption, and welfare at the (possibly stochastic) steady state. Second, we compute the overall welfare impact of access to both public and private storage, including the transition. Third, in Section 4.3, we perform a regression analysis of consumption dynamics in simulated data from our model and the basic model, and compare the way the history of income shocks matters in the two models and in household survey data from village economies. In particular, we use data from three Indian villages collected by the International Crop Research Institute for the Semi-Arid Tropics (ICRISAT).²⁹ Further details on the dataset are in Appendix D.

²⁹It is safe to say that the ICRISAT dataset is the most widely used income-consumption survey from developing countries. In particular, it has been used by many papers studying risk sharing in village economies, including Townsend (1994), Ligon (1998), Ogaki and Zhang (2001), Ligon et al. (2002), Mazzocco and Saini (2012), and Laczó (2015).

4.1 Assets, consumptions, and welfare at the steady state

In this section, we show that public storage can be significant in magnitude. We also illustrate how consumption and welfare are affected by the availability of storage with different returns $-1 \leq r < 1/\beta - 1$ ³⁰ at the (possibly stochastic) steady state.

We assume that agents' per-period utility function is of the CRRA form with a coefficient of relative risk aversion equal to 1, i.e., $u(\cdot) = \ln(\cdot)$. We assume that the income of both agents is i.i.d. over time and may take three values, with equal probabilities. We normalise aggregate income to 1. We match the coefficient of variation of agents' income process to the median coefficient of variation of households' income in the data, which is 0.294. Then, the income values are 0.353, 0.5, and 0.647.

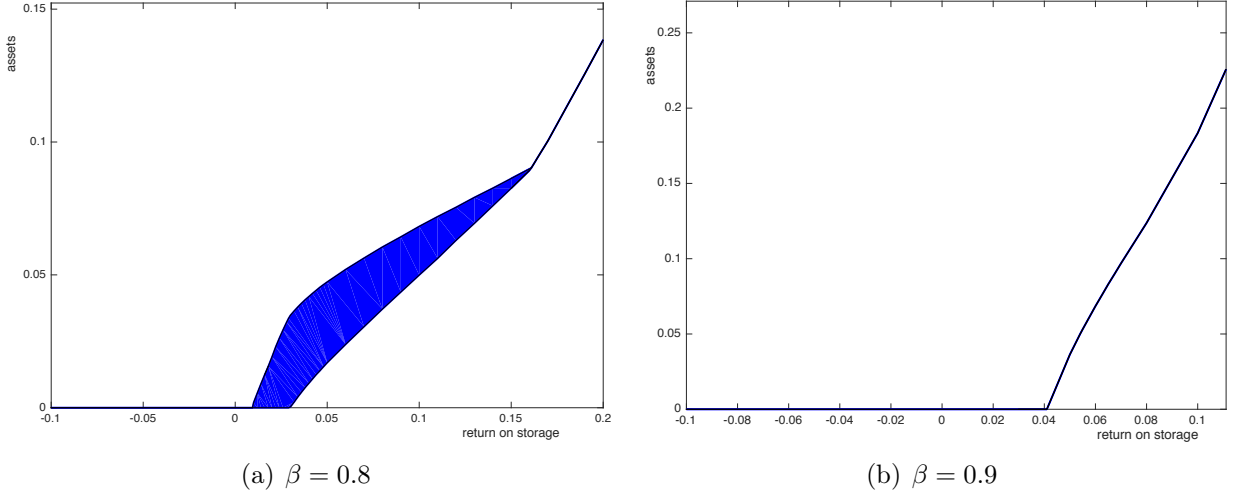
We consider two discount factors, low ($\beta = 0.8$) and high ($\beta = 0.9$). In the former case, risk sharing is partial without storage, however, the consumption distribution is time-varying. In the latter case, perfect risk sharing occurs without access to storage. Note that this does not imply that public and private storage cannot be relevant, as access to private storage increases the autarky values, and may prevent full insurance with zero public assets. This triggers public asset accumulation if the return on storage is sufficiently high.

We present the simulation results in a few figures. First, let us look at the behaviour of assets at the steady state. Figure 2 shows the stationary distribution of assets, the first panel for $\beta = 0.8$ and the second for $\beta = 0.9$. Assets at the steady state naturally increase with r . When the discount factor is high ($\beta = 0.9$), the PCs in state y^m do not bind in the long run, and assets always converge to a constant for any return on storage (case (iii) in Proposition 1). Public storage is strictly positive for $r > 0.041$. For example, with $r = 0.06$ the planner's savings amount to 6.85 percent of aggregate (non-asset) income, with $r = 0.08$ they are 12.37 percent, and with $r = 0.11$ they are at least 22.59 percent.

When $\beta = 0.8$, for intermediate values of r the PCs bind in all three states, and assets remain stochastic at the steady state (case (ii) in Proposition 1). The shaded area in Figure 2, panel (a) indicates the values public storage can take at the steady state for each r , the bounds of which are characterised in Claim 2. Public storage is (sometimes) strictly positive for $r > 0.01$. For example, with $r = 0.06$ public assets vary between 2.39 and 5.20 percent of aggregate (non-asset) income. When the interest rate is $r = 0.02$, assets vary between 0 and 1.91 percent. This last example shows that 0 can be part of the stationary distribution of assets when they are stochastic in the long run (see Claim 2). With $r = 0.25$ public storage

³⁰For $r = 1/\beta - 1$, the FOCA might be invalid, see Appendix C, hence we focus on the case of an inefficient storage technology.

Figure 2: Assets at the steady state



Notes: The stationary distribution of public assets. The aggregate endowment is 1 in each period. Note the difference in scales in the two panels.

reaches at least 21.2 percent of aggregate income.³¹

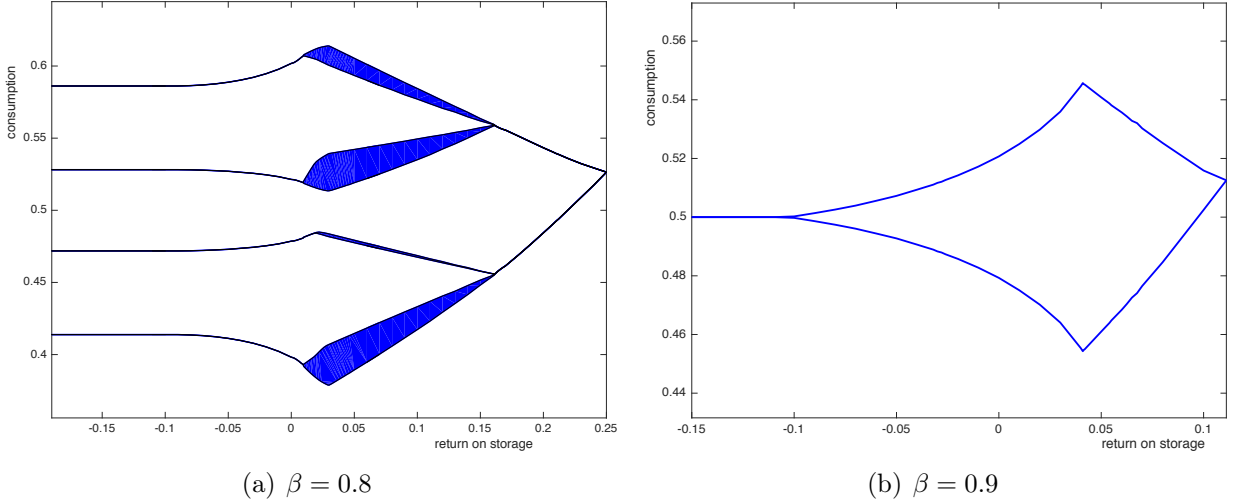
Figure 3 shows the possible steady-state consumption values. Together with Figure 2, this figure reflects the different cases described in Propositions 1 and 6. If $\beta = 0.8$ ($\beta = 0.9$) for returns below $\tilde{r}_1 = -0.091$ ($\tilde{r}_1 = -0.192$) storage does not even affect the value of autarky, and hence it is not used in equilibrium either. In this case, the allocation is not affected by the availability of storage. Given our parameterisation, this implies that in the low-patience case ($\beta = 0.8$) the consumption distribution has four values, while in the high-patience case ($\beta = 0.9$) full risk sharing is enforceable. In fact, for $\beta = 0.9$, perfect risk sharing occurs at the steady state for $r \leq -0.104$, since the PCs still do not bind in the range $-0.192 < r \leq -0.104$. As long as r is below $\tilde{r}_2 = 0.030$ ($\tilde{r}_2 = 0.041$) for β low (high), public storage is still not used, but private storage increases the value of autarky, so consumption dispersion increases with the rate of return on storage.³² For $r \geq \tilde{r}_2$, as r and aggregate asset accumulation increases, consumption dispersion declines.

One important difference between the two cases is that, with the lower β , without public storage a PC binds in state y^m as well. For this reason, in Panel (a) of Figure 3, we see four consumption levels as long as public storage is not used. As the return reaches $r_1 = \tilde{r}_2 = 0.030$, public storage is used, and assets remain stochastic at the steady state until $r_2 = 0.162$ (case (ii) in Proposition 1, see also Figure 2, panel (a)). This implies that in

³¹This is not shown in Figure 2 in order to better highlight the cases where assets are stochastic at the steady state.

³²For $\beta = 0.9$, the autarky value is affected already for a lower storage return, however at these levels full insurance is still enforceable.

Figure 3: Consumption at the steady state

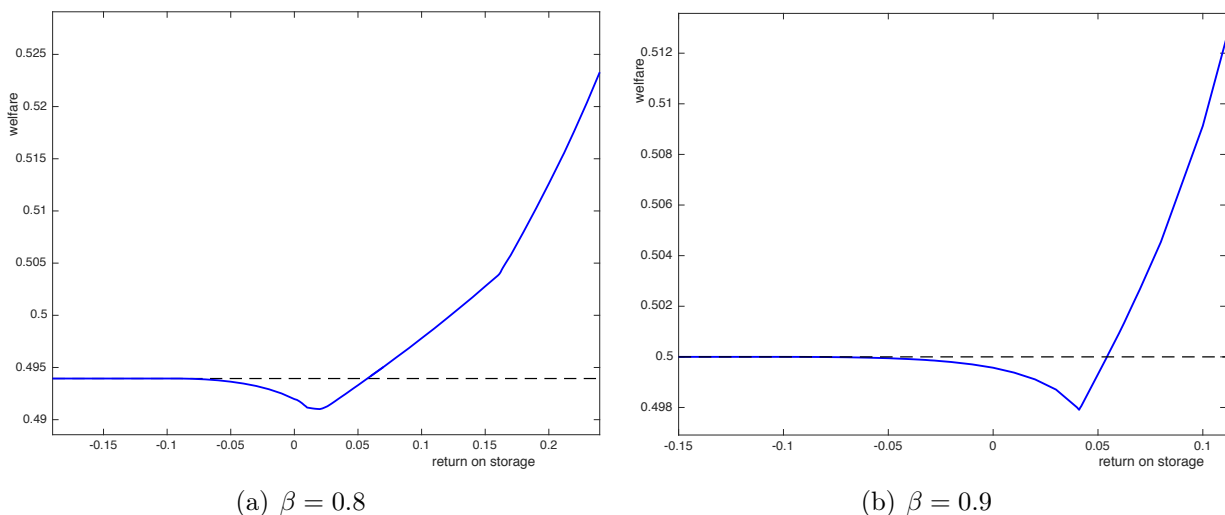


Notes: The stationary distribution of consumption. For $\beta = 0.9$, assets are never stochastic at the steady state and consumption may take two values at most for all r . Note the difference in scales in the two panels.

this case, even in the long run, consumptions depend not only on current income but also on the changing level of assets, and their possible values are indicated by the shaded areas. Panel (a) of Figure 3 also shows that the stochasticity of assets has small but non-negligible effects on the levels and dispersion of consumption in this example. At $r = 0.162$ the PCs stop binding in state y^m , and hence the consumption distribution becomes time-invariant, and assets converge to a constant level.

Figure 4 shows steady-state welfare expressed in per-period consumption. We have characterised steady-state welfare in Proposition 6. When access to storage only increases the value of autarky, it decreases welfare. However, when the return is high enough so that it is used by the planner in equilibrium, it may increase welfare at the steady state. When $\beta = 0.8$ the threshold return above which steady-state welfare improves is $\tilde{r}_3 = 0.057$, when $\beta = 0.9$ it is 0.054. Note that at these thresholds, consumption dispersion is higher than in the case without storage. However, aggregate consumption is also higher. As the return on storage approaches the efficient level, consumption dispersion approaches zero. Hence welfare is always higher with than without storage at the steady state. The welfare gain for r close to the discount rate is approximately equal to a 6.6 percent increase in consumption when $\beta = 0.8$, and to a 2.5 percent increase when $\beta = 0.9$.

Figure 4: Welfare at the steady state



Notes: The solid line shows steady-state welfare in terms of per-period constant consumption with both public and private storage. The dashed line shows steady-state welfare per period in terms of consumption without storage for reference. Note the difference in scales in the two panels.

4.2 Welfare including the transition

Now we compute average welfare from the moment the storage technology becomes available. We do this to take into account the cost of asset accumulation. Figure 5 shows the results. When $\beta = 0.9$ access to both public and private storage lowers welfare for all r . If perfect risk sharing is self-enforcing without private storage (as with $\beta = 0.9$), public storage is never positive even when it is available. This implies that when we allow for private storage, the feasible set shrinks, and hence welfare deteriorates. Panel (b) of Figure 5 confirms this.

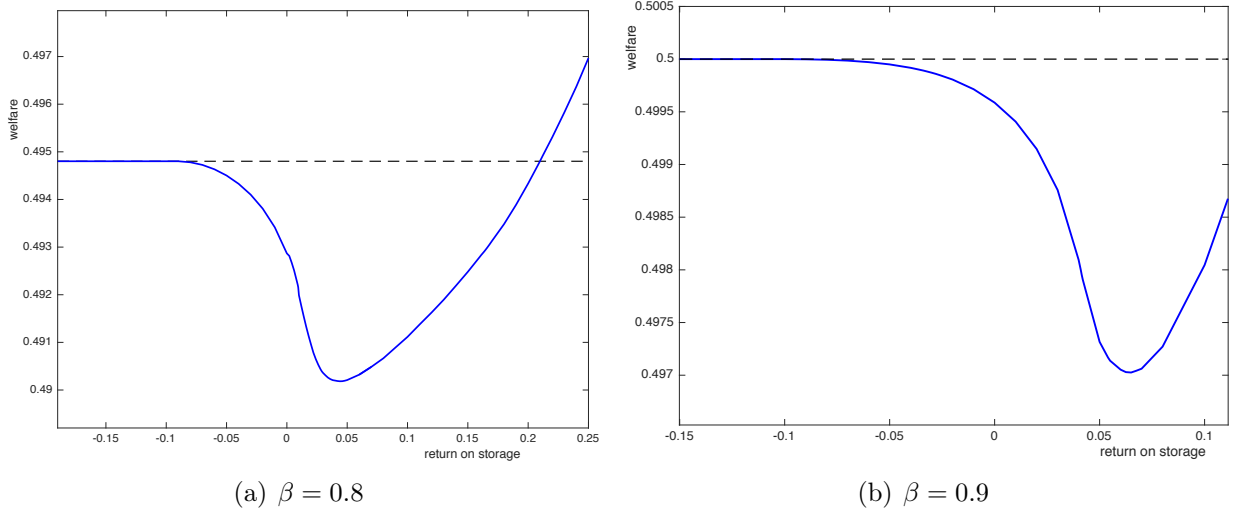
Instead when $\beta = 0.8$ and the return on storage is above the threshold $r = 0.210$, welfare including the transition increases, see Panel (a) of Figure 5. Here risk sharing is partial without private storage, and the overall effect could go either way. There are welfare costs associated with the build-up of aggregate assets, but the steady-state gains dominate when the return on storage is sufficiently high.

4.3 Consumption dynamics – model and village data

We want to examine whether the dynamics of consumption in our model are more similar to that of household survey data compared to the basic model. In particular, we study the effect of past income shocks.

To derive an estimating equation, consider the usual optimality condition that the ratio of marginal utilities between any two agents/households, i and $-i$, should equal the current

Figure 5: Welfare including transition



Notes: The solid line shows expected lifetime utility in terms of per-period constant consumption from the moment when (both public and private) storage becomes available. The dashed line shows expected lifetime utility per period in terms of consumption without storage for reference. Note the difference in scales in the two panels.

relative Pareto weight. That is, with CRRA preferences, $(c_{-i,t})^{-\sigma} / (c_{it})^{-\sigma} = x_{it}$. Taking logs and rearranging, we have

$$\ln c_{it} - \ln c_{-i,t} = \frac{\ln x_{it}}{\sigma}.$$

Under perfect risk sharing, $\ln x_{it} = \ln x_{i,t-1}$. Typical tests of perfect risk sharing in the vein of the seminal paper of [Townsend \(1994\)](#) use past consumptions to capture the ratio of marginal utilities, and establish whether new information at time t , such as income realisations, affect consumptions. Then, the estimating equation can be written as

$$\ln c_{it} - \ln c_{-i,t} = \alpha_0 + \alpha_1 (\ln c_{i,t-1} - \ln c_{-i,t-1}) + \alpha_2 (\ln y_{it} - \ln y_{-i,t}) + u_{it}, \quad (22)$$

where u_{it} is an error term. A α_2 significantly different from zero suggests that perfect risk sharing fails.

The main motivation to introduce LC in the risk-sharing literature was to explain α_2 significantly positive when one estimates (22), using data from village economies, for example. According to the basic LC model, $\ln x_{it} = f(\ln x_{i,t-1}, \mathbf{y}_t)$, where \mathbf{y}_t is the vector of incomes at time t and $f()$ is some function. Controlling for $\ln x_{i,t-1}$, \mathbf{y}_t , and approximating the function $f()$ sufficiently well, past information such as past incomes should not affect current consumptions significantly. That is, in the regression

$$\ln c_{it} - \ln c_{-i,t} = \alpha_0 + f(\ln x_{i,t-1}, \mathbf{y}_t) + \gamma (\ln y_{i,t-1} - \ln y_{-i,t-1}) + u_{it}, \quad (23)$$

where γ should not be significantly different from zero according to the basic model. Instead, according to our model with public storage, γ might be different from zero, because of time-varying public storage. We are also interested in whether the sign of γ when estimating (23) using real data and model-simulated data is the same.

To investigate the empirical relevance of the two LC models, without and with storage, in accounting for consumption dynamics, we estimate (23) using

- (i) simulated data from the basic LC model,
- (ii) simulated data from our LC model with storage, and
- (iii) actual data from the ICRISAT villages.

For our simulated data $i = 1$ and $-i = 2$, while for the village data $-i$'s are village means in equation (23).³³

To implement the regression on simulated data, we first have to calibrate our models, and generate data which have sufficient variation in the variables of interest for our regression. In order to do this, we first estimate an AR(1) process for the (logarithm of) incomes of four types of households in each of the three villages.³⁴ We then approximate each estimated AR(1) process by a 4-state Markov chain, following Tauchen (1986) and Kennan (2006). Finally, we normalise total income to 1 and make both the income states and the transition matrix symmetric.³⁵ In addition, we allow for three household types in terms of impatience: β can take three values, 0.8, 0.825, and 0.85. Note that time preferences are homogeneous for a pair of agents sharing risk. This implies that we end up with 36 types of pairs (3 villages \times 4 income process types \times 3 impatience types).³⁶ We assume that the utility function is of the CRRA form with a coefficient of relative risk aversion equal to 1.5. This means that inverse marginal utility is strictly convex, hence public storage increases with inequality. We use data for 50 periods for each pair, once the long-run distribution has been reached (after 100 periods in practice). For our model with storage we have to choose the return on storage as well. We pick $r = 0.1$.

³³We do not know of a satisfactory treatment of the N -household case in limited commitment models, and structural empirical studies of risk sharing in village economies follow the household vs. rest of the village approximation, see Ligon, Thomas, and Worrall (2000), Laczó (2015), and Bold and Broer (2016). We follow this procedure for the data.

³⁴The four groups are created based on whether a household's mean income and the coefficient variation of its income is below or above the median, as in Laczó (2015), which yields 4 household types per village. Note that we cannot estimate a household-specific income process, as we observe each household for only 6 periods.

³⁵Note that only small adjustments are necessary, as the distribution of the logarithm of incomes is close to symmetric in the data.

³⁶To achieve a large simulated sample with sufficient variation, simulating one couple over many periods does not work, as eventually one gets repeated observations on the variables of interest.

Finally, we have to approximate the function $f(\ln x_{i,t-1}, \mathbf{y}_t)$. We include both linear and non-linear terms of the difference in the previous period's consumptions and in current incomes.³⁷ In particular, we include the square and the cube of our control variables.³⁸ Then we end up with the following estimating equation:

$$\begin{aligned} \ln c_{it} - \ln c_{-i,t} = & \alpha_0 + \alpha_1 (\ln c_{i,t-1} - \ln c_{-i,t-1}) + \alpha_2 (\ln y_{it} - \ln y_{-i,t}) + \alpha_3 (\ln c_{i,t-1} - \ln c_{-i,t-1})^2 \\ & + \alpha_4 (\ln y_{it} - \ln y_{-i,t})^2 + \alpha_5 (\ln c_{i,t-1} - \ln c_{-i,t-1})^3 \\ & + \alpha_6 (\ln y_{it} - \ln y_{-i,t})^3 + \gamma (\ln y_{i,t-1} - \ln y_{-i,t-1}) + u_{it}. \end{aligned} \quad (24)$$

The results are presented in Table 1. They show that introducing storage in the LC model is a step in the right direction to account for the dependence of current consumption shares on past income shares. Past income shares are insignificant in the basic model, confirming the result of the literature. Instead in our model and in the data, past income shares have a significant negative effect. Note, however, that alternative models of partial risk sharing may also result in a negative coefficient on past incomes. Comparing the performance of our model in terms of consumption dynamics with that of other settings with different frictions is beyond the scope of this paper.

Table 1: Dependent variable: $\ln c_{it} - \ln c_{-i,t}$

Coefficient	no storage	$r = 0.1$	data
α_0	0.005 (0.115)	0.005 (0.287)	0.031 (0.061)
α_1	0.481 (0.000)	0.504 (0.000)	0.735 (0.000)
α_2	0.197 (0.000)	0.304 (0.000)	0.211 (0.000)
α_3	-0.098 (0.284)	0.019 (0.764)	-0.250 (0.056)
α_4	-0.005 (0.096)	0.002 (0.596)	-0.002 (0.935)
α_5	0.391 (0.132)	0.146 (0.617)	-0.294 (0.032)
α_6	-0.087 (0.000)	-0.061 (0.000)	0.021 (0.163)
γ	-0.001 (0.863)	-0.017 (0.010)	-0.071 (0.018)
R^2	0.625	0.784	0.661
# of obs.	1800	1800	510

Note: p -values are in parentheses.

The negative coefficient on past incomes in our model is the consequence of history dependence due to asset accumulation. In order to gain intuition, note that by controlling for $\ln c_{i,t-1} - \ln c_{-i,t-1}$ in (24), we control for x_{t-1} . At the same time, we do not control for the

³⁷A linear regression may be insufficient to well approximate the function $f()$, and as a result, past income might be significant due to the persistence of the income process even in the basic model, where yesterday's ratio of marginal utilities and current incomes are all the state variables.

³⁸We have experimented with several alternative specifications. Including only the square of the control variables is not sufficient to capture the non-linearity of $f()$ in the basic model.

level of assets/storage. However, the current level of storage is a function of x_{t-1} and B_{t-1} . This implies that by controlling for $\ln c_{i,t-1} - \ln c_{-i,t-1}$, we also control for the ‘part’ of assets which is explained by x_{t-1} . Hence, a significant coefficient on $\ln y_{i,t-1} - \ln y_{-i,t-1}$ comes from the contemporaneous correlation between public assets and income inequality at time $t - 1$. Because of the persistence of incomes, current income inequality and income inequality of the previous period are positively correlated. Further, under convex inverse marginal utility, higher income inequality implies more storage (see Claim 1). Therefore, public assets and income inequality are positively correlated. Now, given that B_t is ceteris paribus increasing in B_{t-1} , and that consumption inequality is decreasing in current public assets, we have that consumption differences at time t are negatively correlated with income differences at time $t - 1$.

5 Summary and discussion

This paper has shown that some implications of the basic LC model with no private or public storage are not robust to hidden storage. Instead when public storage is allowed, the incentive for private storage is eliminated in the constrained-efficient allocation. The intertemporal technology is used in equilibrium even though the aggregate endowment is constant and the return is lower than the discount rate. Further, when income inequality is not the highest, the planner has more incentive to store than the agents. The reason for additional storage by the planner is that public assets relax future PCs and hence improve risk sharing.

The effects of the availability of both public and private storage on asset accumulation, consumption dispersion, and welfare depend on its return. At the steady state, (i) for low r , access to storage is welfare neutral, because it is not used, hence we are back to the basic model of Kocherlakota (1996); (ii) for higher r , storage happens only in autarky, therefore, consumption dispersion increases and welfare decreases, but storage does not matter otherwise; (iii) for yet higher r , hidden storage matters in equilibrium in the basic model, public storage is (sometimes) strictly positive, stochastic, and depends positively on consumption inequality (as long as inverse marginal utility is convex), the consumption distribution is time-varying, and many consumption values occur;³⁹ (iv) for yet higher r , public storage becomes positive and constant at the steady state, and only two consumption levels occur, i.e., the consumption distribution is time-invariant; (v) for $r = 1/\beta - 1$, public storage is positive and constant, and perfect risk sharing occurs. Steady-state welfare improves above some

³⁹This third case only occurs for some set of parameter values.

threshold return as a result of access to both public and private storage. For low β welfare including the transition (i.e., taking into account the cost of accumulating assets) improves as well above a higher threshold return, which is less than the discount rate.

The dynamics of individual consumptions are richer in our model compared to the basic model when assets are stochastic at the steady state. In particular, the amnesia and persistence properties do not hold in general, as in the data (Kinnan, 2017; Broer, 2013). We provide regression evidence that past incomes have a negative impact on current consumptions in our model, consistent with data from village economies. This is the result of a positive correlation between past incomes and storage when inverse marginal utility is convex. Further, in our model agents' Euler inequalities hold for a wide range of storage returns, which is consistent with empirical evidence from developed countries (Attanasio, 1999).

Comparing our model with LC and storage to models with hidden income or effort and storage (Cole and Kocherlakota, 2001) points to some similarities and remarkable differences. In both models hidden storage has a welfare-reducing effect, as it impose tighter constraints on risk sharing. In private information models, public storage cannot mitigate this effect and hence is never used in equilibrium. In contrast, in our model public storage is used in equilibrium. This is because with LC as the deep friction, storage by the planner relaxes the incentive problem, by relaxing future PCs, while in the hidden income/effort context, aggregate asset accumulation makes incentive provision for truthful revelation more costly.

Throughout the analysis, we have restricted our attention to a model without aggregate risk. We have done this to isolate the effect of LC on asset accumulation from other motives, such as aggregate consumption smoothing. However, we expect our key results to hold with aggregate income risk as well. Clearly, if the return on storage is high enough, public assets would be accumulated and would fluctuate even at the first best. It is well known from the literature with exogenous incomplete markets (Huggett, 1993, Aiyagari, 1994) that assets are bounded as long as $\beta(1+r) < 1$.⁴⁰ This implies that when one combines aggregate income risk and LC, assets will be stochastic and bounded at the steady state.

The main mechanism of our paper would operate in a very similar way in the presence of aggregate risk. In fact, the key equation determining public asset accumulation, the planner's Euler equation, (8), would remain the same except that the histories would include the realisations of the aggregate shock as well. This also means that, as in our model without aggregate risk, the introduction of public storage relaxes the market friction, unlike in incomplete market models with asymmetric information. Further, if aggregate income is uncorrelated with cross-sectional income inequality, this implies that, compared to the

⁴⁰It would not be the case under $\beta(1+r) = 1$, hence we would expect assets to diverge with LC as well.

first best, the constrained-efficient allocation would exhibit more asset accumulation. This is because it is not only helpful for aggregate consumption smoothing, but also for reducing future consumption inequality, as without aggregate risk. If aggregate income is negatively correlated with cross-sectional income inequality, a potentially empirically relevant case, the two forces determining aggregate asset accumulation go in opposite directions, and we would expect smoother asset behaviour than at the first best.

As both the private and public Euler equations remain virtually unchanged, the introduction of aggregate risk would not affect another key result either: public storage preempts private storage in equilibrium. This result is particularly useful for solving the model numerically, which would therefore be without difficulty in the presence of aggregate risk in quantitative applications. In terms of welfare, as storage has an intrinsic value even without the LC friction with aggregate risk, we expect that the higher aggregate risk is, the more likely the overall welfare effect of access to storage is positive.

Our model could be applied in several economic contexts. The model predicts that risk sharing among households in villages can be improved by a public grain storage facility. Our model also provides a rationale for marriage contracts to specify that some commonly held assets are lost by the spouse who files for divorce. Finally, supranational organisations may help international risk sharing by simply having a jointly held stock of assets. The European Stability Mechanism may serve this purpose. Future work should study the quantitative implications of storage using some of these applications.

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Appendices

A Proofs

Proof of Claim 1. We consider three income states for expositional reasons. Generalising the proof to more income states is straightforward. Assume indirectly that $B'(B, \tilde{x}') = B'(B, \hat{x}') \equiv B'$.⁴¹ This assumption and (9) imply that $u'(c(B, \tilde{x}')) < u'(c(B, \hat{x}'))$.

First, consider \tilde{x}' and \hat{x}' such that $\min\{\underline{x}^h(B'), \bar{x}^m(B')\} \geq \tilde{x}' > \hat{x}' \geq 1$. Let us rewrite (11) as

$$1 \geq \beta(1+r) \sum_{y'} \Pr(y') \frac{u'(C(y', B', x'))}{u'(c(B, x'))(1-v_1(y', B', x'))}. \quad (25)$$

We now detail what happens next period, so that we can compare the right-hand side of (25) for \tilde{x}' and \hat{x}' .

- If state y^h occurs, then the PC of agent 1 is binding. Given that B' is the same for \tilde{x}' and \hat{x}' under our indirect assumption, x'' will equal $\underline{x}^h(B')$ and c' will equal $\underline{c}^h(B')$ for both. However, the ratio on the right-hand side of (25) differs because $v_1(y', B', \tilde{x}') < v_1(y', B', \hat{x}')$. For $x' = \{\hat{x}', \tilde{x}'\}$ we obtain

$$\frac{u'(\underline{c}^h(B'))}{u'(c(B, x'))(1-v_1(y', B', x'))} = \frac{u'(\bar{c}^l(B'))}{u'(c_2(B, x'))}$$

where we have combined (9) and (10).

- If state y^m occurs, then no PC is binding, hence the relative Pareto weight does not change. For HARA utility functions, it can be shown using simple algebra that each agent's marginal utility grows at the rate $((2a + c' + c'_2)/(2a + c + c_2))^{-\sigma}$, hence we know that in this case

$$\frac{u'(C(y^m, B', \tilde{x}'))}{u'(c(B, \tilde{x}'))} = \frac{u'(C(y^m, B', \hat{x}'))}{u'(c(B, \hat{x}'))}.$$

- If state y^l occurs, then the PC of agent 2 is binding. Given that B' is the same for \tilde{x}' and \hat{x}' , x'' will equal $\bar{x}^l(B')$ and c' will equal $\bar{c}^l(B')$ for both. Thus for $x' = \{\hat{x}', \tilde{x}'\}$, we have

$$\frac{u'(\bar{c}^l(B'))}{u'(c(B, x'))}.$$

In summary, for $x' = \{\hat{x}', \tilde{x}'\}$ on the right-hand side of (25) we have

$$\beta(1+r) \left[\pi^e \frac{u'(\bar{c}^l(B'))}{u'(c_2(B, x'))} + \pi^m \frac{u'(C(y^m, B', x'))}{u'(c(B, x'))} + \pi^e \frac{u'(\bar{c}^l(B'))}{u'(c(B, x'))} \right],$$

⁴¹If we assume indirectly that $B'(B, \tilde{x}') \leq B'(B, \hat{x}')$ for $\tilde{x}' > \hat{x}' \geq 1$ and $B'(B, \tilde{x}') \geq B'(B, \hat{x}')$ for $1 \geq \tilde{x}' > \hat{x}'$, the steps of the proof are the same, but the algebra is more tedious.

where $\pi^e = \pi^h = \pi^l$. If this expression is greater for \tilde{x}' than for \hat{x}' , then our indirect assumption is invalidated and B' has to be greater for \tilde{x}' than for \hat{x}' to satisfy (25). The second term is the same in the two expressions. Therefore, the sign of the difference is the sign of

$$\Delta_1(B, \tilde{x}', \hat{x}') \equiv \frac{1}{u'(c_2(B, \tilde{x}'))} + \frac{1}{u'(c(B, \tilde{x}'))} - \left(\frac{1}{u'(c_2(B, \hat{x}'))} + \frac{1}{u'(c(B, \hat{x}'))} \right).$$

Given that $\tilde{x}' > \hat{x}' \geq 1$ implies $c_2(B, \tilde{x}') < c_2(B, \hat{x}') \leq c(B, \hat{x}') < c(B, \tilde{x}')$ by Property 1, this difference is (strictly) positive if $1/u'$ is (strictly) convex. So under this condition, B' is (strictly) increasing in x' in the case where $\min\{\underline{x}^h(B'), \bar{x}^m(B')\} \geq x' \geq 1$.

Second, consider \tilde{x}' and \hat{x}' such that $\tilde{x}' > \hat{x}' \geq \bar{x}^m$.

- If state y^l occurs next period, nothing changes compared to the previous case, where $\min\{\underline{x}^h(B'), \bar{x}^m(B')\} \geq \tilde{x}' > \hat{x}' \geq 1$.
- For state y^m the difference between the ratio on the right-hand side of (25) for \tilde{x}' and \hat{x}' is

$$\Delta_2(B, B', \tilde{x}', \hat{x}') \equiv \frac{u'(\bar{c}^m(B'))}{u'(c(B, \tilde{x}'))} - \frac{u'(\bar{c}^m(B'))}{u'(c(B, \hat{x}'))} > 0.$$

- In state y^h three cases are possible.
 - The PC of agent 1 is binding for both \tilde{x}' and \hat{x}' . Then we can use the previous case. Note that the difference between the right-hand side of (25) for \tilde{x}' and \hat{x}' is given by $\Delta_1(B, \tilde{x}', \hat{x}') + \Delta_2(B, B', \tilde{x}', \hat{x}') > 0$.
 - The PC of agent 1 is not binding for either x' . Then the growth rate of marginal utility is the same for \tilde{x}' and \hat{x}' . In this case, the difference between the right-hand side of (25) for \tilde{x}' and \hat{x}' is given by

$$\pi^e u'(\bar{c}^l(B')) \left(\frac{1}{u'(c(B, \tilde{x}'))} - \frac{1}{u'(c(B, \hat{x}'))} \right) + \Delta_2(B, B', \tilde{x}', \hat{x}') > 0.$$

- The PC of agent 1 is binding for \hat{x}' , but not for \tilde{x}' . Then $c_2(B', \hat{x}') < \bar{c}^l(B')$. Therefore, the difference between the right-hand sides of (25) for \tilde{x}' and \hat{x}' is given by

$$\Delta_1(B, \tilde{x}', \hat{x}') + \Delta_2(B, B', \tilde{x}', \hat{x}') + \frac{u'(c_2(B', \hat{x}')) - u'(\bar{c}^l(B'))}{u'(c_2(B, \tilde{x}'))} > 0.$$

Finally, consider \tilde{x}' and \hat{x}' such that $\tilde{x}' \geq \bar{x}^m(B') > \hat{x}'$. The only difference compared to the previous case is in state y^m . We have $c(B', \hat{x}') < \bar{c}^m(B')$. This implies that

$$\Delta_3(B, B', \tilde{x}', \hat{x}') = \frac{u'(\bar{c}^m(B'))}{u'(c(B, \tilde{x}'))} - \frac{u'(c(B', \hat{x}'))}{u'(c(B, \hat{x}'))} > 0.$$

Hence the same argument as in the previous case follows replacing $\Delta_2(B, B', \tilde{x}', \hat{x}')$ with $\Delta_3(B, B', \tilde{x}', \hat{x}')$.

Since the problem is symmetric, to establish the relationship between B' and $x' \leq 1$, we can consider $1/x' \geq 1$. This means that B' increases as $x' \leq 1$ decreases, i.e., as cross-sectional consumption inequality increases.

If $j > k$, and the optimal intervals for these two states do not overlap given B , then x' must be higher in state y^j than in state y^k , and we have already shown that assets depend positively on cross-sectional consumption inequality. If the optimal intervals overlap given B , then there exists x for which $x' = x$ in both states y^j and y^k . Aggregate savings are identical in the two states in this case. \square

Proof of Proposition 1. Part (i). It is easy to see that r_1 is implicitly defined by the planner's Euler, (11), as equality when agent 1 has the highest possible income. That is, r_1 is implicitly given by

$$u'(\mathcal{C}(y^J, 0, \underline{x}^J(0))) = \beta(1 + r_1) \sum_j \pi^j \frac{u'(\mathcal{C}(y^j, 0, \underline{x}^J(0)))}{1 - v_1(y^j, 0, \underline{x}^J(0))}.$$

If $r > r_1$ public assets will be positive at least when income inequality is highest, while if $r \leq r_1$ public assets will be zero in the long run, and will always be zero if their initial level is zero.

Next, we show that assets are bounded, which we need for parts (ii)-(iv). They are trivially bounded below by 0. It is easy to see that there exists a high level of assets, denoted \widehat{B} , such that perfect risk sharing is at least temporarily enforceable, that is, $\bar{x}^1(\widehat{B}) \geq \underline{x}^J(\widehat{B})$. Therefore, if $r < 1/\beta - 1$, $B'(B, x') < B$ for all $B \geq \widehat{B}$ and $\bar{x}^1(B) \geq x' \geq \underline{x}^J(B)$, i.e., assets optimally decrease; and assets stay constant if $r = 1/\beta - 1$. This implies that assets are bounded above.

We now turn to parts (ii) and (iii). We first show that if the consumption distribution is time-invariant, then there exists a unique constant level of assets, B^* , such that all the conditions of constrained-efficiency are satisfied. Afterwards, we show that assets converge almost surely to B^* starting from any initial level B_0 . Then, we establish that assets remain stochastic when the consumption distribution is time-varying (case (ii)). Finally, we show that case (iii) occurs when the return on storage is high but less than the discount rate, while assets remain stochastic when the return is below some threshold, denoted r_2 .

Recall that if aggregate assets are constant, the optimal intervals for the relative Pareto weight are time-invariant. Given that each agent's PC binds only for the highest income level in the long run, the optimality condition (9) and $\underline{x}^J(B^*)$ ($\bar{x}^1(B^*)$) uniquely determine $\underline{c}^J(B^*)$ ($\bar{c}^1(B^*)$), the time-invariant high (low) consumption level. Then, using the planner's Euler, we can determine the unique level of B^* such that all optimality conditions are satisfied. The planner's Euler is

$$u'(\underline{c}^J(B^*)) = \beta(1 + r) [(1 - \pi^e) u'(\underline{c}^J(B^*)) + \pi^e u'(\bar{c}^1(B^*))],$$

where $\pi^e = \pi^J = \pi^1$. Dividing both sides by $u'(\underline{c}^J(B^*))$, we obtain

$$\begin{aligned} 1 &= \beta(1+r) \left[(1-\pi^e) + \pi^e \frac{u'(c^1(B^*))}{u'(\underline{c}^J(B^*))} \right] \\ &= \beta(1+r) \left[(1-\pi^e) + \pi^e \underline{x}^J(B^*) \right], \end{aligned} \tag{26}$$

where we have used (9). Note that $\underline{x}^J(B^*)$ is monotone and continuous in B^* . Further, at $B^* = 0$ the right-hand side of equation (26) is larger than 1 by assumption, and at $B^* = \widehat{B}$ the right-hand side of (26) is smaller than 1, because $\underline{x}^J(\widehat{B}) = 1$ and $B^* < \widehat{B}$. Therefore, we know that there exists a unique B^* where the planner's Euler holds as equality by setting $B' = B = B^*$.

Next, we show that assets converge almost surely to B^* starting from any initial level, B_0 . We already know that $B'(B_0, x') < B_0$ for the ergodic range of x' when $B_0 \geq \widehat{B}$, i.e., when perfect risk sharing is (temporarily) self-enforcing, and $B'(0, x') > 0$ for some x' in the ergodic range of x' , since $r > r_1$ by assumption. Consider $B^* < B_0 < \widehat{B}$ first, and assume that state y^J occurs and agent 1's PC is binding. This is without loss of generality, because this happens with probability 1 in the long run, and the problem is symmetric across the two agents. We know that the right-hand side of (26) is smaller than 1, because $\underline{x}^J(B_0) < \underline{x}^J(B^*)$. Therefore, marginal utility tomorrow has to increase relative to marginal utility today to satisfy the planner's Euler, hence $B'(B_0, \underline{x}^J(B_0)) < B_0$. What happens next period? The PC will bind again even if the same state occurs.⁴² This is because $B'(B_0, \underline{x}^J(B_0)) < B_0$ implies $\underline{x}^J(B'(B_0, \underline{x}^J(B_0))) > \underline{x}^J(B_0)$. Then assets will decrease again. What if some state y^j with $2 \leq j \leq N-1$ occurs? We know that the PCs in these states are not binding for any $B \geq B^*$, because they are not binding for B^* . This means that now $x' = x = \underline{x}^J(B_0) < \underline{x}^J(B'(B_0, \underline{x}^J(B_0)))$. Then, by Claim 1, storage is lower than when the PC is binding. Note that if states $\{y^2, \dots, y^{J-1}\}$ occur repeatedly, assets converge to a level below B^* . Then we are in the case where $B_0 < B^*$, which we now turn to.

Consider $0 \leq B_0 < B^*$, and suppose again that state y^J occurs and agent 1's PC is binding. Then we know that $\underline{x}^J(B_0) > \underline{x}^J(B^*)$. Using (26) again, it follows that $B'(B_0, \underline{x}^J(B_0)) > B_0$. Now, if the same state occurs next period (in fact, any state y^j with $j \geq 2$), then the PC is slack. This means that now $x' = x = \underline{x}^J(B_0) > \underline{x}^J(B'(B_0, \underline{x}^J(B_0)))$. Then, by Claim 1, storage is higher than when the PC is binding. This also implies that if state y^1 does not occur for many periods, assets converge to a level above B^* . Then once y^1 occurs, which happens with probability 1 in the long run, we are back to the case $B_0 > B^*$, and assets start decreasing.⁴³

So far we have shown that when $B_0 < B^*$, assets increase at least in the most unequal states. Unless we are on a path where agents get the highest income shock exactly in turns,

⁴²Note that this never happens in the basic model.

⁴³PCs in more states may be binding when B is low, even if they only bind in states y^1 and y^J for B^* . However, with probability 1 assets will reach a level where the PCs of the other states are no longer binding.

assets converge toward a level higher than B^* . We have also shown that whenever $B_0 > B^*$ and an agent's PC binds, assets decrease. Again, unless one of the agents always receives the highest shock, assets converge to a value lower than B^* . This implies that assets oscillate around B^* . Almost sure convergence is guaranteed because these oscillations shrink whenever a PC binds in the increasing and/or decreasing part, which happens with probability one. To see this, note that from Claim 1 we know that $B'(B, \underline{x}^J(B_1))$ is highest when $\underline{x}^J(B_1)$ is highest. In turn, $\underline{x}^J(B_1)$ is highest when B_1 is lowest. That is, the economy might get close to the highest possible B during the transition if starting with zero public assets state y^1 (or y^J) keeps occurring. Similarly, once we are above B^* , the lowest possible level of B can be reached with a most equal state occurring infinitely many times, if that state starts occurring when assets are at their highest possible level. Note that the upper bound and the lower bound are reached with probability zero. Whenever there is a switch to y^1 or y^J , we get closer to B^* . Then the new possible highest asset level is lower and the lowest asset level is higher (and again the bounds are reached with probability zero). Then, again, the 'circle' shrinks when there is a switch to y^1 or y^J .

Part (ii). Consider the case where at the steady state there is a third state in which a PC binds. In this case, each agent's consumption takes at least four different values at the steady state. These have to satisfy an additional PC, an additional resource constraint, and an additional Euler, which is generically impossible for constant B .

Finally, we have to show that case (ii) occurs if $r_1 < r \leq r_2$, while case (iii) occurs if $r_2 < r < 1/\beta - 1$. It is easy to see that B^* is lower if r is lower, where B^* can be computed for any r ignoring the PCs of states y^j with $2 \leq j \leq N - 1$. However, as assets decrease, the optimal intervals become narrower, and eventually $\bar{c}^2(B) < \underline{c}^J(B)$ and $\bar{c}^1(B) < \underline{c}^{J-1}(B)$. Hence, r_2 is implicitly given by (26) such that B^* is such that $\bar{x}^2(B^*) = \underline{x}^J(B^*)$ (and $\bar{x}^1(B^*) = \underline{x}^{J-1}(B^*)$).

Part (iv). If risk sharing were imperfect in the next period, then it would be that $\underline{c}^J(B') > \bar{c}^1(B')$. Then, from the planner's Euler, (8), with $\beta(1+r) = 1$ we have that $\underline{c}^J(B') > \underline{c}^J(B)$, which implies that $B' > B$. That is, public assets are increasing. This means that as long as a PC binds given B , the planner has an incentive to store more. Hence, at the steady state B is constant as in case (iii), and risk sharing is perfect. \square

Proof of Lemma 2. We first show that if c' is weakly increasing in x'' next period, then c is strictly increasing in x' in the current period using Claim 1. Given $\tilde{x}' > \hat{x}'$, six cases are possible in terms of the pattern of binding PCs next period in a given income state. Depending on the number of income states, the width of the optimal intervals, and \tilde{x}' and \hat{x}' , not all these types of states necessarily exist.

- (i) The PC of agent 1 is binding for both \tilde{x}' and \hat{x}' in state y' next period.⁴⁴ Let $\tilde{x}''(y') \equiv x''(y', B'(B, \tilde{x}'), \tilde{x}')$, and similarly for $\hat{x}''(y')$, $\tilde{x}''(Y - y')$, and $\hat{x}''(Y - y')$.

⁴⁴Clearly, if \tilde{x}' and \hat{x}' are sufficiently high, there will be no such y' .

Given $B'(B, \tilde{x}') > B'(B, \hat{x}')$, we know that $1 < \tilde{x}' < \tilde{x}''(y') = \underline{x}^{y'}(B'(B, \tilde{x}')) < \underline{x}^{y'}(B'(B, \hat{x}')) = \hat{x}''(y')$, which implies $y' > Y/2$. Then, $\tilde{x}' > \hat{x}'$ and (10) imply that

$$\frac{1}{1 - v_1(y', B'(B, \tilde{x}'), \tilde{x}')} < \frac{1}{1 - v_1(y', B'(B, \hat{x}'), \hat{x}')},$$

because x has to increase more from \hat{x}' to \hat{x}'' than from \tilde{x}' to \tilde{x}'' . Now, by symmetry, there is also a state $Y - y' < Y/2$ next period, which occurs with the same probability as state y' . We will show that the consumption allocation next period for this pair of states under current Pareto weight \tilde{x}' has a lower spread and a higher mean than the allocation under current Pareto weight \hat{x}' . For this we have to consider whether PCs bind in state $Y - y'$ next period.

- First, assume that $\tilde{x}' > \bar{x}^{Y-y'}(B'(B, \tilde{x}'))$ and $\hat{x}' > \bar{x}^{Y-y'}(B'(B, \hat{x}'))$, i.e., the PC of agent 2 is binding in state $Y - y'$ for both \tilde{x}' and \hat{x}' . Then, by symmetry, $\tilde{x}''(Y - y') = 1/\tilde{x}''(y') > 1/\hat{x}''(y') = \hat{x}''(Y - y')$.
- Second, assume that $\tilde{x}' \leq \bar{x}^{Y-y'}(B'(B, \tilde{x}'))$ and $\hat{x}' \leq \bar{x}^{Y-y'}(B'(B, \hat{x}'))$, i.e., no PC is binding in state $Y - y'$ for either \tilde{x}' or \hat{x}' . Then, $\tilde{x}''(Y - y') = \tilde{x}' > \hat{x}' = \hat{x}''(Y - y')$.
- Third, assume that $\tilde{x}' > \bar{x}^{Y-y'}(B'(B, \tilde{x}'))$ and $\hat{x}' \leq \bar{x}^{Y-y'}(B'(B, \hat{x}'))$, i.e., the PC of agent 2 is binding for \tilde{x}' but not for \hat{x}' . It follows that $\tilde{x}''(Y - y') = \tilde{x}' > \bar{x}^{Y-y'}(B'(B, \tilde{x}')) > \bar{x}^{Y-y'}(B'(B, \hat{x}')) = \hat{x}''(Y - y')$, where the second inequality holds because $B'(B, \tilde{x}') > B'(B, \hat{x}')$ and the optimal intervals are wider when assets are greater.
- Fourth, assume that $\tilde{x}' \leq \bar{x}^{Y-y'}(B'(B, \tilde{x}'))$ and $\hat{x}' > \bar{x}^{Y-y'}(B'(B, \hat{x}'))$, i.e., the PC of agent 2 is binding for \hat{x}' but not for \tilde{x}' . It follows that $\tilde{x}''(Y - y') = \bar{x}^{Y-y'}(B'(B, \tilde{x}')) \geq \tilde{x}' > \hat{x}' = \hat{x}''(Y - y')$.

In all four cases $\hat{x}''(y') \geq \tilde{x}''(y') > \tilde{x}''(Y - y') \geq \hat{x}''(Y - y')$, hence the consumption allocation given \tilde{x}' has a smaller spread across the states y' and $Y - y'$. It also has a higher mean, because of the higher level of assets and a lower x'' , which implies less storage next period by Claim 1 as long as $\hat{x}''(y') \geq \tilde{x}''(y') \geq 1$, which must be the case here. As the mean decreases, expected marginal utility increases. What happens to expected marginal utility as a result of a higher spread? Under prudence, the marginal utility function is decreasing and convex, therefore, expected marginal utility is higher for the more risky process. Finally, the term $1/(1 - v_1(\cdot))$ further increases the right-hand side of (11) given \hat{x}' relative to \tilde{x}' , which implies that c is strictly increasing in x' even if c' is only weakly increasing in x'' .

- (ii) The PC of agent 1 is binding for \hat{x}' but not for \tilde{x}' in state y' next period. In this case, either $\tilde{x}''(y') \geq \hat{x}''(y')$ or $\tilde{x}''(y') < \hat{x}''(y')$. If $\tilde{x}''(y') \geq \hat{x}''(y')$ consumption next period is higher for \tilde{x}' , because of a higher current Pareto weight and more resources than

for \hat{x}' . This implies a lower marginal utility tomorrow for \tilde{x}' . In addition, once again the term $1/(1 - v_1())$ further increases the right-hand side of (11) given \hat{x}' relative to \tilde{x}' . If $\tilde{x}''(y') < \hat{x}''(y')$, then we can use the same argument as in case (i). Since $\tilde{x}''(y') = \tilde{x}' > \underline{x}^{y'}(B'(B, \tilde{x}'))$, expected marginal utility next period is yet lower given \tilde{x}' for this reason.

(iii) No PC is binding for \tilde{x}' or \hat{x}' next period. In this case, consumption next period is strictly higher for \tilde{x}' than for \hat{x}' because of a higher B' , so marginal utility next period is strictly lower for \tilde{x}' than for \hat{x}' , and both $1/(1 - v_1())$ s are 1.

(iv)-(vi) The PC of agent 2 is binding for \tilde{x}' , or for \hat{x}' , or for both next period. In these cases, we can use similar arguments as above to show that $\tilde{x}''(y') > \hat{x}''(y')$, and hence consumption next period is strictly higher for \tilde{x}' than for \hat{x}' .

In all six types of states (or pairs of states), the right-hand side of (11) is strictly lower for \tilde{x}' than for \hat{x}' , therefore the left-hand side must be strictly lower as well. This means that c must be strictly higher when x' is higher, given that c' depends positively on x'' .

Proposition 1 shows that assets converge to a constant level at the steady state almost surely if r is higher than some threshold r_2 . That is, at the steady state the characteristics of allocations are the same as in the basic model (while aggregate consumption is $Y + rB^*$ rather than Y), in particular, c strictly increases with x' . Then, moving backwards in time, c must strictly increase with x' in all periods.

Finally, we know that the solution is unique, therefore we can conclude that it is characterised by the consumption of agent 1 increasing in x' for all r such that assets are constant at the steady state. \square

Proof of Claim 2. From Claim 1 it is clear that \underline{B} is approached if a least unequal income state, denoted y^m , happens repeatedly, while \overline{B} is approached with state y^J (or y^1) happening many times in a row.

If B is part of the stationary distribution, then it must be that $B \geq \underline{B}$. This means that there are less and less resources available over time while assets approach \underline{B} , hence the relevant PC always binds along this path. The planner's Euler

$$u'(\bar{c}^m(\underline{B})) \geq \beta(1+r) [\pi^e u'(\mathcal{C}(y^l, \underline{B}, \bar{x}^m(\underline{B}))) + (1 - 2\pi^e) u'(\bar{c}^m(\underline{B})) + \pi^e u'(\mathcal{C}(y^h, \underline{B}, \bar{x}^m(\underline{B})))]$$

as equality defines \underline{B} if $\underline{B} > 0$. If at $\underline{B} = 0$ this Euler is satisfied as a strict inequality, then the lower bound is 0.

The upper limit of the stationary distribution, \overline{B} , is approached from below, hence, along that path, the highest shock (state y^J or y^1) happens repeatedly and no PC binds. Let B_1 denote the level of assets when we switch to state y^J (or y^1), and let \tilde{B} denote the level of

assets to where B converges. Note that along this path the relative Pareto weight is constant at $\underline{x}^J(B_1)$. Given B_1 , \tilde{B} is the solution to the following system:

$$\frac{u' \left(\mathcal{C}_2 \left(y^J, \tilde{B}, \underline{x}^J(B_1) \right) \right)}{u' \left(\mathcal{C} \left(y^J, \tilde{B}, \underline{x}^J(B_1) \right) \right)} = \underline{x}^J(B_1)$$

$$\mathcal{C} \left(y^J, \tilde{B}, \underline{x}^J(B_1) \right) + \mathcal{C}_2 \left(y^J, \tilde{B}, \underline{x}^J(B_1) \right) = Y + r\tilde{B}$$

$$u' \left(\mathcal{C} \left(y^J, \tilde{B}, \underline{x}^J(B_1) \right) \right) = \beta(1+r) \sum_{j=1}^J \pi^j u' \left(\mathcal{C} \left(y^j, \tilde{B}, \underline{x}^j(B_1) \right) \right). \quad (27)$$

We have to find B_1 such that \tilde{B} is equal to \bar{B} , the upper limit of the stationary distribution. Using Claim 1, we know that $B'(B, \underline{x}^J(B_1))$ is highest when $\underline{x}^J(B_1)$ is highest. In turn, $\underline{x}^J(B_1)$ is highest when B_1 is lowest, i.e., when B_1 is equal to the lower limit of the stationary distribution of assets, \underline{B} . Then, replacing $\underline{x}^J(B_1)$ with $\underline{x}^J(\underline{B})$ and \tilde{B} with \bar{B} in (27) gives (17). \square

Proof of Proposition 6. (i) It is easy to see that storage is never used when its return is close to -1 , i.e., as long as it is below some threshold \tilde{r}_1 . (ii) It is similarly easy to see that storage in equilibrium implies storage in autarky. This follows from the fact that the planner's and the agents' saving incentives are the same when income inequality is highest, i.e., when the incentive to store is highest, and agents' Euler inequality is more stringent in autarky than in equilibrium with some risk sharing. Then, if storage only takes place in autarky, the only effect of storage is that the value of agents' outside option increases, which reduces risk sharing and welfare. However, the value of autarky does not matter as long as perfect risk sharing is self-enforcing, hence, as long as that is the case, access to storage is welfare neutral. (iii) As r further increases to above the threshold \tilde{r}_2 , according to Proposition 1 the planner finds public storage optimal. However, by continuity, at this point the negative effect of the increase in the value of autarky dominates the positive effect of the (small) stock of public assets on risk sharing. Therefore, welfare still goes down as a result of access to storage. (iv)-(v) If $r = 1/\beta - 1$, perfect risk sharing occurs and aggregate consumption is $Y + rB^*$ rather than Y , therefore welfare is strictly higher at the steady state. Further, consumption dispersion is zero. Then, for any r in a small neighbourhood of $1/\beta - 1$, the positive effect of the increase in aggregate consumption dominates the negative effect of the increase in the value of autarky, hence welfare improves. For such r , consumption dispersion is small. By continuity there exists $\tilde{r}_3 \in (\tilde{r}_2, 1/\beta - 1)$, where the two welfare levels are equalised. At this level of storage return, aggregate consumption has to be higher than in the basic model (at least after some histories). Hence, welfare can be the same only if consumption dispersion is higher than in the basic model. By continuity this should hold above \tilde{r}_3 as well until some threshold $\tilde{r}_4 \leq 1/\beta - 1$. \square

B Computation

We use the recursive system given by equations (9)-(14) to solve the model numerically. We discretise x and B (y is assumed to take a finite number of values). We have to determine x' and B' on a 3-dimensional grid on $X = (y, B, x)$. The initial values for $\mathcal{V}(X')$, $\mathcal{C}(X')$, and $v_1(X')$ are from the solution of a model where the PCs are ignored. We iterate until the value and policy functions converge.

As we proceed, we use the characteristics of the solution. In particular, we know that if agent 1's PC binds at \tilde{x} , it also binds at all $x < \tilde{x}$. Similarly, if agent 2's PC binds at \hat{x} , it also binds at all $x > \hat{x}$. At each iteration, at each income state, and for each B , we solve directly for the limits \tilde{x} and \hat{x} using (12) and (13) as equality, respectively, first assuming that $B' = 0$. Afterwards, we check whether the planner's Euler is satisfied at the limits. If not, we solve a 2-equation system of (11) and (12) (or (13)), with unknowns B' and x' . Finally, we solve for a new B' at points on the x grid where neither PC binds, i.e., at the interior of the optimal interval for (y, B) of the current iteration. We linearly interpolate future policy and value functions.

C Validity of the first-order-conditions approach

In Section 3 we have assumed that by introducing agents' PCs and Euler inequalities (equations (18) and (19), respectively) in Problem *P1*, we guarantee incentive compatibility. In other words, we have assumed that the constrained-optimal allocation can be obtained by checking that agents have no incentive to default given that they do not have private assets, and that they have no incentive to store in a hidden way given that they never default. In principle, they may still find it optimal to use more complicated 'double' deviations involving both storage and default, potentially in different time periods, given some history of income shocks. In this appendix we discuss how to verify that such double deviations are not beneficial, i.e., that the first-order-conditions approach is valid. Our approach is similar to that of Werning (2002) and Ábrahám and Pavoni (2008).

First, note that we have already considered contemporaneous joint deviations, i.e., when the agent defaults and saves at the same time.⁴⁵ In the PC (18) we use $\tilde{U}_i^{au}(s_t)$, the value of autarky when the agent can store privately (see equation (20)). Further, note that in autarky the agent is allowed to store privately whenever this makes him better off. Therefore, the 'default today and store later'-type of double deviations are already taken care of as well. This implies that the only potentially profitable double deviations we still need to consider

⁴⁵In the literature with private information, a similar joint deviation, shirking (or misreporting income) and saving, is the relevant deviation. Detailed discussion of these joint deviations can be found for the hidden income case in Cole and Kocherlakota (2001), and for the hidden action (dynamic moral hazard) case in Kocherlakota (2004) and Ábrahám, Koehne, and Pavoni (2011).

are those which involve private asset accumulation in the current period and default in a later period.

We can show analytically that as long as the constrained-efficient consumption values do not exceed the autarky consumption values when the agent holds no private assets, ‘default today and store later’-type of double deviations are never profitable for the agents.⁴⁶ However, for high values of r , the highest possible consumption value in equilibrium is higher than the autarky consumption value in that state.⁴⁷ In fact, [Nozawa \(2016\)](#) shows, in a one-sided LC framework, that there exists a profitable deviation in the case where $\beta(1+r) = 1$, and this deviation involves saving in one period and defaulting in the next. Note that the deviation happens when the economy has reached its long-run equilibrium and perfect risk sharing occurs, and that the agent’s PC holds as equality when he gets the highest possible income. In our model these conditions are satisfied only when public assets converge to their lowest possible steady-state value given an efficient storage technology, denoted \underline{B}^* , a zero-probability event. However, given that the double deviation is strictly beneficial for the agent, in a small joint neighbourhood of $(B, r) = (\underline{B}^*, 1/\beta - 1)$ the double deviation is still beneficial by continuity.

We provide a numerical algorithm to verify whether ‘store first and default later’-type of double deviations are welfare-improving. We find that agents have no incentive to use ‘store first and default later’-type of double deviations outside this small neighbourhood. In particular, we show numerically that given any level of public assets, incomes, and the inherited relative Pareto weight, agents are better off receiving as endowment the consumptions assigned by the constrained-efficient risk-sharing contract rather than their own incomes today or in the future. In order to see this, along with the autarky consumption-saving problem, we solve the consumption-saving problem of an agent who receives the constrained-efficient consumption process as ‘income.’ We take into account that if an agent has accumulated savings \tilde{b} at the end of a period, which he has saved out of his constrained-efficient consumption process, his outside option next period when his income is z is $V^{au}(z, \tilde{b})$, which is defined in (20). Having computed the constrained-efficient policy functions as described in Appendix B, this is without conceptual difficulty, however, the computational cost is rather high, given that there are four state variables, three of which are continuous. We again exploit the characteristics of the solution, namely that the current Pareto weight takes values within an optimal state-dependent interval, in order to shorten computation time.

In examples we have studied, where (B, r) is not in a small neighbourhood of $(\underline{B}^*, 1/\beta - 1)$, we find that agents are always better off receiving the constrained-efficient consumptions, given any level of public assets, rather than the autarky incomes. Hence, they will never

⁴⁶The proof is available upon request.

⁴⁷In the long run, this only happens for returns ‘close’ to the efficient level, but during the transition this may happen for returns below the threshold r_2 in Proposition 1 as well. This is easy to verify in computed examples.

revert to autarky and will never store in a hidden fashion, as long as the first-order conditions are satisfied.

D Data

We use data from the Village Level Studies conducted by the International Crop Research Institute for the Semi-Arid Tropics (ICRISAT) in India from 1975 to 1984.⁴⁸ We focus on three villages, Aurepalle, Kanzara, and Shirapur, and the years 1976 to 1981, because of concern over the accuracy of measured consumption in the other years (Townsend, 1994; Ligon, Thomas, and Worrall, 2002). Our non-durable consumption measure includes food consumption, clothing, services, utilities, and narcotics. Income includes net income from crop production, labour, livestock, and transfers from outside the village. Both consumption and income used in the analysis are yearly and per adult equivalent. To compute the adult-equivalent size of each household, we use the same age-gender weights as Townsend (1994).⁴⁹

Table 2 presents descriptive statistics for the three villages. On average, daily non-durable consumption per adult equivalent is 0.83, 1.10, and 1.18 1975 Indian rupees in Aurepalle, Kanzara, and Shirapur, respectively, which is about 0.42, 0.55, and 0.59 2016 US dollars, respectively.

Table 2: Descriptive statistics

Variable	Aurepalle		Kanzara		Shirapur	
	Mean	Sd	Mean	Sd	Mean	Sd
Aeq. non-dur. cons. ^a	303.47	127.86	400.84	161.42	430.37	170.71
Aeq. income ^a	629.58	429.78	984.42	742.54	792.16	577.57
# of observations	204		222		186	
# of households	34		37		31	

^aMeasured in 1975 Indian rupees per year. In 1975, approximately 8 Indian rupees were worth 1 US dollar, which is about 4 dollars in 2016.

⁴⁸We thank ICRISAT for making the data available, Reena Badiani and Ethan Ligon for making their constructed aggregates available, and Maurizio Mazzocco and Shiv Saini for sharing data construction codes.

⁴⁹These weights are: 1 for adult males, 0.9 for adult females, 0.94 and 0.83 for males and females aged 13-18, respectively, 0.67 for children aged 7-12, 0.52 for children aged 4-6, 0.32 for children aged 1-3, and 0.05 for infants below 1 year of age.